

Elliptic Quantum Group $U_{q,p}(B_N^{(1)})$ and Vertex Operators

Hitoshi Konno[†] and Kazuyuki Oshima^{*}

[†]*Department of Mathematics, Tokyo University of Marine Science and Technology,
Etchujima, Koto, Tokyo 135-8533, Japan
hkonno0@kaiyodai.ac.jp*

^{*}*Center for General Education, Aichi Institute of Technology,
Yakusa-cho, Toyota 470-0392, Japan
oshima@aitech.ac.jp*

Dedicated to Professor Akihiro Tsuchiya on his 70th birthday.

Abstract

Assuming the existence of the L -operators, we study the Hopf algebroid structure of $U_{q,p}(B_N^{(1)})$. As an application, we derive the type I and II vertex operators, which intertwine the $U_{q,p}(B_N^{(1)})$ -modules of generic level, by assuming some analytic properties of the L -operators. For the level-1 case, we construct their free field realizations and show that the results satisfy the desired commutation relations with coefficients given by the elliptic dynamical R -matrices of the $B_N^{(1)}$ type.

1 Introduction

The algebra $U_{q,p}(\widehat{\mathfrak{g}})$ [6, 13, 20] is an elliptic analogue of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization [4] associated with the affine Lie algebra $\widehat{\mathfrak{g}}$. The $U_{q,p}(\widehat{\mathfrak{g}})$ is expected to give a realization of the face type elliptic quantum group [7, 12] equipped with the Hopf algebroid structure. In the previous works [13, 18, 23], we have constructed the L -operator of $U_{q,p}(A_N^{(1)})$ in terms of the elliptic currents, the generating functions of the Drinfeld generators of $U_{q,p}(A_N^{(1)})$. The L -operator satisfies the RLL -relation with the elliptic dynamical R -matrix of the $A_N^{(1)}$ type [10] and allows us to define the Hopf algebroid structure to $U_{q,p}(A_N^{(1)})$.

The elliptic quantum group $U_{q,p}(A_N^{(1)})$ equipped with the Hopf algebroid structure has proved to be quite useful in construction of both the finite and infinite dimensional representations as well as their intertwining operators, i.e. the vertex operators, in terms of the free fields. Such construction becomes a central tool in the algebraic analysis of the face type solvable lattice

models associated with the vector representation of $\widehat{\mathfrak{g}}$ [10] in the spirit of Jimbo and Miwa [11]. See for example [19, 26, 27].

The purpose of this paper is to continue the above study to the case $U_{q,p}(B_N^{(1)})$. The $U_{q,p}(B_N^{(1)})$ itself has an interesting connection to the deformation of Fateev-Lukyanov's [25] WB_N algebra [6]. Assuming the existence of the L -operators in the elliptic algebra $U_{q,p}(B_N^{(1)})$, we give an Hopf algebroid structure of $U_{q,p}(B_N^{(1)})$. We then define the type I and II vertex operators as the intertwining operators of the $U_{q,p}(B_N^{(1)})$ -modules of generic level. By assuming some analytic properties of the L -operators and the half currents, which are expected to be defined recursively through the Gauss decomposition of the L -operators, we show that the components of the both types of vertex operators are constructed by applying certain half currents to the top component. For the level-1 case, we construct their free field realizations and show that the results satisfy the desired commutation relations with coefficients given by the elliptic dynamical R -matrices of the $B_N^{(1)}$ type. These results give elliptic and dynamical analogues of those obtained for $U_q(B_N^{(1)})$ in [1, 14].

This paper is organized as follows. In Sec.2, we define the elliptic algebra $U_{q,p}(B_N^{(1)})$ as a certain topological algebra. In particular, we introduce the orthonormal basis type elliptic bosons and define the elliptic currents $k_{\pm j}(z)$. The Sec.3 is devoted to a conjecture of the construction of the L -operators in terms of the half currents of $U_{q,p}(B_N^{(1)})$. In Sec.4, assuming the existence of the L -operators we introduce the H -Hopf algebroid structure to $U_{q,p}(B_N^{(1)})$. In Sec.5, we give the vector representation and the level-1 highest weight representation of $U_{q,p}(B_N^{(1)})$. In Sec.6, after giving a construction of the type I and II vertex operators at generic level, we present a free field realization of the level 1 vertex operators and show that they satisfy the desired commutation relations with the coefficients given by the elliptic dynamical R -matrices. In Appendix A, we summarize a connection of $U_{q,p}(B_N^{(1)})$ to the quasi-Hopf formulation $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ of the elliptic quantum group. In Appendix B, we give a list of conjectural expressions for the half currents of $U_{q,p}(B_N^{(1)})$.

2 Elliptic Algebra $U_{q,p}(B_N^{(1)})$

In this section, we give a definition of the elliptic algebra $U_{q,p}(B_N^{(1)})$ associated with the affine Lie algebra $B_N^{(1)}$.

2.1 Definition

Let $A = (a_{ij})$ $i, j \in \{0\} \cup I$, $I = \{1, \dots, N\}$ be the $B_N^{(1)}$ type generalized Cartan matrix [15]. We denote by $B = (b_{ij})$, $b_{ij} = d_i a_{ij}$ the symmetrization of A with $d_0 = \dots = d_{N-1} = 1, d_N = 1/2$. Let $q = e^h \in \mathbb{C}[[\hbar]]$ and set $q_i = q^{d_i}$. Let p be an indeterminate. We use the following notations.

$$\begin{aligned} [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i = \frac{q^n - q^{-n}}{q_i - q_i^{-1}}, \\ [n]_i! &= [n]_i [n-1]_i \cdots [1]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[n]_i! [m-n]_i!}, \\ (x; q)_\infty &= \prod_{n=0}^{\infty} (1 - xq^n), \quad (x; q, t)_\infty = \prod_{n,m=0}^{\infty} (1 - xq^n t^m), \quad \Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty. \end{aligned}$$

Let $\mathfrak{h} = \tilde{\mathfrak{h}} \oplus \mathbb{C}d$, $\tilde{\mathfrak{h}} = \bar{\mathfrak{h}} \oplus \mathbb{C}c$, $\bar{\mathfrak{h}} = \oplus_{i \in I} \mathbb{C}h_i$ be the Cartan subalgebra of $B_N^{(1)}$. Define $\delta, \Lambda_0, \alpha_i$ ($i \in I$) $\in \mathfrak{h}^*$ by

$$\langle \alpha_i, h_j \rangle = a_{ji}, \quad \langle \delta, d \rangle = 1 = \langle \Lambda_0, c \rangle, \quad (2.1)$$

the other pairings are 0. We also define $\bar{\Lambda}_i$ ($i \in I$) $\in \mathfrak{h}^*$ by

$$\langle \bar{\Lambda}_i, h_j \rangle = \delta_{i,j} \quad (2.2)$$

We set $\bar{\mathfrak{h}}^* = \oplus_{i \in I} \mathbb{C}\bar{\Lambda}_i$, $\tilde{\mathfrak{h}}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$, $\mathcal{Q} = \oplus_{i \in I} \mathbb{Z}\alpha_i$ and $\mathcal{P} = \oplus_{i \in I} \mathbb{Z}\bar{\Lambda}_i$. Let $\{\epsilon_j$ ($1 \leq j \leq N$) $\}$ be an orthonormal basis in \mathbb{R}^N with the inner product $(\epsilon_j, \epsilon_k) = \delta_{j,k}$. We realize the simple roots by $\alpha_j = \epsilon_j - \epsilon_{j+1}$ ($1 \leq j \leq N-1$), $\alpha_N = \epsilon_N$ and the fundamental weights by $\bar{\Lambda}_j = \epsilon_1 + \dots + \epsilon_j$ ($1 \leq j \leq N-1$), $\bar{\Lambda}_N = (\epsilon_1 + \dots + \epsilon_N)/2$. We define $h_{\epsilon_j} \in \bar{\mathfrak{h}}$ ($j \in I$) by $\langle \epsilon_i, h_{\epsilon_j} \rangle = (\epsilon_i, \epsilon_j)$ and $h_\alpha \in \bar{\mathfrak{h}}$ for $\alpha = \sum_j c_j \epsilon_j$, $c_j \in \mathbb{C}$ by $h_\alpha = \sum_j c_j h_{\epsilon_j}$. We regard $\bar{\mathfrak{h}} \oplus \bar{\mathfrak{h}}^*$ as the Heisenberg algebra by

$$[h_{\epsilon_j}, \epsilon_k] = (\epsilon_j, \epsilon_k), \quad [h_{\epsilon_j}, h_{\epsilon_k}] = 0 = [\epsilon_j, \epsilon_k]. \quad (2.3)$$

In particular, we have $[h_j, \alpha_k] = a_{jk}$. We also set $h^j = h_{\bar{\Lambda}_j}$.

Let $\{P_\alpha, Q_\beta\}$ ($\alpha, \beta \in \bar{\mathfrak{h}}^*$) be the Heisenberg algebra defined by the commutation relations

$$[P_{\epsilon_j}, Q_{\epsilon_k}] = (\epsilon_j, \epsilon_k), \quad [P_{\epsilon_j}, P_{\epsilon_k}] = 0 = [Q_{\epsilon_j}, Q_{\epsilon_k}], \quad (2.4)$$

where $P_\alpha = \sum_j c_j P_{\epsilon_j}$ for $\alpha = \sum_j c_j \epsilon_j$. We set $P_{\bar{\mathfrak{h}}} = \oplus_{j \in I} \mathbb{C}P_{\epsilon_j}$, $Q_{\bar{\mathfrak{h}}} = \oplus_{j \in I} \mathbb{C}Q_{\epsilon_j}$, $P_j = P_{\alpha_j^\vee}$, $P^j = P_{\bar{\Lambda}_j}$ and $Q_j = Q_{\alpha_j}$, $Q^j = Q_{\bar{\Lambda}_j^\vee}$.

For the abelian group $\mathcal{R}_Q = \sum_{j=1}^N \mathbb{Z}Q_{\alpha_j}$, we denote by $\mathbb{C}[\mathcal{R}_Q]$ the group algebra over \mathbb{C} of \mathcal{R}_Q . We denote by e^α the element of $\mathbb{C}[\mathcal{R}_Q]$ corresponding to $\alpha \in \mathcal{R}_Q$. These e^α satisfy $e^\alpha e^\beta = e^{\alpha+\beta}$ and $(e^\alpha)^{-1} = e^{-\alpha}$. In particular, $e^0 = 1$ is the identity element.

Now let us consider to double the Cartan subalgebra : $H = \tilde{\mathfrak{h}} \oplus P_{\tilde{\mathfrak{h}}} = \sum_j \mathbb{C}(P_{\tilde{\epsilon}_j} + h_{\tilde{\epsilon}_j}) + \sum_j \mathbb{C}P_{\tilde{\epsilon}_j} + \mathbb{C}c$. We denote its dual space by $H^* = \tilde{\mathfrak{h}}^* \oplus Q_{\tilde{\mathfrak{h}}}$. We define the paring by (2.2), $\langle Q_\alpha, P_\beta \rangle = (\alpha, \beta)$ and $\langle Q_\alpha, h_\beta \rangle = \langle Q_\alpha, c \rangle = \langle Q_\alpha, d \rangle = 0 = \langle \alpha, P_\beta \rangle = \langle \delta, P_\beta \rangle = \langle \Lambda_0, P_\beta \rangle$. We define $\mathbb{F} = \mathcal{M}_{H^*}$ to be the field of meromorphic functions on H^* .

Definition 2.1. [6] The elliptic algebra $U_{q,p}(B_N^{(1)})$ is a topological algebra over $\mathbb{F}[[p]]$ generated by \mathcal{M}_{H^*} , $e_{j,m}, f_{j,m}, \alpha_{j,n}^\vee, K_j^\pm$, ($j \in I, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}$), \hat{d} and the central element c . We assume K_j^\pm are invertible and set

$$\begin{aligned} e_j(z) &= \sum_{m \in \mathbb{Z}} e_{j,m} z^{-m}, \quad f_j(z) = \sum_{m \in \mathbb{Z}} f_{j,m} z^{-m}, \\ \psi_j^+(q^{-\frac{c}{2}} z) &= K_j^+ \exp \left(-(q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,-n}^\vee}{1-p^n} z^n \right) \exp \left((q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}^\vee}{1-p^n} z^{-n} \right), \\ \psi_j^-(q^{\frac{c}{2}} z) &= K_j^- \exp \left(-(q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,-n}^\vee}{1-p^n} z^n \right) \exp \left((q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,n}^\vee}{1-p^n} z^{-n} \right). \end{aligned}$$

We call $e_j(z), f_j(z), \psi_j^\pm(z)$ the elliptic currents. The defining relations are as follows. For $g(P), g(P+h) \in \mathcal{M}_{H^*}$,

$$g(P+h)e_j(z) = e_j(z)g(P+h), \quad g(P)e_j(z) = e_j(z)g(P - \langle Q_{\alpha_j}, P \rangle), \quad (2.5)$$

$$g(P+h)f_j(z) = f_j(z)g(P+h - \langle \alpha_j, P+h \rangle), \quad g(P)f_j(z) = f_j(z)g(P), \quad (2.6)$$

$$[g(P), \alpha_{i,m}^\vee] = [g(P+h), \alpha_{i,n}^\vee] = 0, \quad (2.7)$$

$$g(P)K_j^\pm = K_j^\pm g(P - \langle Q_{\alpha_j}, P \rangle), \quad (2.8)$$

$$g(P+h)K_j^\pm = K_j^\pm g(P+h - \langle Q_{\alpha_j}, P \rangle), \quad (2.9)$$

$$[\hat{d}, g(P+h)] = [\hat{d}, g(P)] = 0, \quad (2.10)$$

$$[\hat{d}, \alpha_{j,n}^\vee] = n\alpha_{j,n}^\vee, \quad [\hat{d}, e_j(z)] = -z \frac{\partial}{\partial z} e_j(z), \quad [\hat{d}, f_j(z)] = -z \frac{\partial}{\partial z} f_j(z), \quad (2.11)$$

$$K_i^\pm e_j(z) = q_i^{\mp a_{ij}} e_j(z) K_i^\pm, \quad K_i^\pm f_j(z) = q_i^{\pm a_{ij}} f_j(z) K_i^\pm, \quad (2.12)$$

$$[\alpha_{i,m}^\vee, \alpha_{j,n}^\vee] = \delta_{m+n,0} \frac{[a_{ij}m]_i [cm]_j}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (2.13)$$

$$[\alpha_{i,m}^\vee, e_j(z)] = \frac{[a_{ij}m]_i}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e_j(z), \quad (2.14)$$

$$[\alpha_{i,m}^\vee, f_j(z)] = -\frac{[a_{ij}m]_i}{m} z^m f_j(z), \quad (2.15)$$

$$z_1 \frac{(q^{b_{ij}} z_2 / z_1; p^*)_\infty}{(p^* q^{-b_{ij}} z_2 / z_1; p^*)_\infty} e_i(z_1) e_j(z_2) = -z_2 \frac{(q^{b_{ij}} z_1 / z_2; p^*)_\infty}{(p^* q^{-b_{ij}} z_1 / z_2; p^*)_\infty} e_j(z_2) e_i(z_1), \quad (2.16)$$

$$z_1 \frac{(q^{-b_{ij}} z_2 / z_1; p)_\infty}{(pq^{b_{ij}} z_2 / z_1; p)_\infty} f_i(z_1) f_j(z_2) = -z_2 \frac{(q^{-b_{ij}} z_1 / z_2; p)_\infty}{(pq^{b_{ij}} z_1 / z_2; p)_\infty} f_j(z_2) f_i(z_1), \quad (2.17)$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-c} z_1 / z_2) \psi_j^-(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi_j^+(q^{-\frac{c}{2}} z_2) \right), \quad (2.18)$$

$$\begin{aligned}
& \sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(p^* q^2 z_{\sigma(k)} / z_{\sigma(m)}; p^*)_{\infty}}{(p^* q^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p^*)_{\infty}} \\
& \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(p^* q^{b_{ij}} w / z_{\sigma(m)}; p^*)_{\infty}}{(p^* q^{-b_{ij}} w / z_{\sigma(m)}; p^*)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{b_{ij}} z_{\sigma(m)} / w; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_{\sigma(m)} / w; p^*)_{\infty}} \\
& \times e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(s)}) e_j(w) e_i(z_{\sigma(s+1)}) \cdots e_i(z_{\sigma(a)}) = 0, \tag{2.19}
\end{aligned}$$

$$\begin{aligned}
& \sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(pq^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p)_{\infty}}{(pq^2 z_{\sigma(k)} / z_{\sigma(m)}; p)_{\infty}} \\
& \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(pq^{-b_{ij}} w / z_{\sigma(m)}; p)_{\infty}}{(pq^{b_{ij}} w / z_{\sigma(m)}; p)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(pq^{-b_{ij}} z_{\sigma(m)} / w; p)_{\infty}}{(pq^{b_{ij}} z_{\sigma(m)} / w; p)_{\infty}} \\
& \times f_i(z_{\sigma(1)}) \cdots f_i(z_{\sigma(s)}) f_j(w) f_i(z_{\sigma(s+1)}) \cdots f_i(z_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}), \tag{2.20}
\end{aligned}$$

where $p^* = pq^{-2c}$ and $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. We also denote by $U'_{q,p}(\widehat{\mathfrak{g}})$ the subalgebra obtained by removing d .

We treat the relations (2.11), (2.14)-(2.20) as formal Laurent series in z, w and z_j 's. All the coefficients in z_j 's are well-defined in the p -adic topology.

2.2 The orthonormal basis type elliptic bosons

It is convenient to introduce the simple root type generators $\alpha_{j,m}$ and $\alpha'_{j,m}$ defined by $\alpha_{j,m} = [d_j]_q \alpha_{j,m}^{\vee}$ and $\alpha'_{j,m} = \frac{1 - p^{*m}}{1 - p^m} q^{cm} \alpha_{j,m}$, ($j \in I, n \neq 0$). From (2.13), (2.14), (2.15), we have

$$[\alpha_{i,m}, \alpha_{j,n}] = \frac{[b_{ij}m]_q [cm]_q}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-km} \delta_{m+n,0}, \tag{2.21}$$

$$[\alpha'_{i,m}, \alpha'_{j,n}] = \frac{[b_{ij}m]_q [cm]_q}{m} \frac{1 - p^{*m}}{1 - p^m} q^{km} \delta_{m+n,0}, \tag{2.22}$$

$$[\alpha_{i,m}, \alpha'_{j,n}] = \frac{[b_{ij}m]_q [cm]_q}{m} \delta_{m+n,0}, \tag{2.23}$$

$$[\alpha_{i,m}, e_j(z)] = \frac{[b_{ij}m]_q}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} z^m e_j(z), \tag{2.24}$$

$$[\alpha'_{i,m}, f_j(z)] = -\frac{[b_{ij}m]_q}{m} \frac{1 - p^{*m}}{1 - p^m} q^{cm} z^m f_j(z). \tag{2.25}$$

Let $\eta = -(2N - 1)/2$. Let us further introduce the orthonormal basis type elliptic bosons $\mathcal{E}_m^{\pm j}$ ($j \in \{0\} \cup I, m \in \mathbb{Z}_{\neq 0}$) [6] by

$$\mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left(q^{\pm \eta m} \sum_{k=1}^{j-1} [km]_q \alpha_{k,m} \pm \sum_{k=j}^N [(\eta + k)m]_+ \alpha_{k,m} \right), \tag{2.26}$$

$$\mathcal{E}_m^0 = \frac{[\frac{m}{2}]_q}{[m]_q} (\mathcal{E}_m^{+N} + \mathcal{E}_m^{-N}). \tag{2.27}$$

Here we set

$$C_m = \frac{[\eta m]_q}{[m]_q^2 [2\eta m]_q}, \quad [m]_+ = \frac{q^m + q^{-m}}{q - q^{-1}}.$$

Proposition 2.2.

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] = \delta_{m+n,0} \frac{[cm]_q [\eta m]_q [2(\eta+1)m]_q}{m(q-q^{-1})^2 [m]_q^3 [2\eta m]_q [(\eta+1)m]_q} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (2.28)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] = \mp \delta_{m+n,0} \frac{q^{\pm jm} [cm]_q [\eta m]_q}{m [m]_q^3 (q-q^{-1})^2 [2\eta m]_q} \frac{1-p^m}{1-p^{*m}} q^{-cm} \left(q^{\pm(\eta+j)m} [m]_q \pm q^{\mp(j-1)m} [\eta m]_+ \right), \quad (2.29)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] = \mp \text{sgn}(k-j) \delta_{m+n,0} q^{\mp(\text{sgn}(k-j)\eta+k-j)m} \frac{[cm]_q [\eta m]_q}{m(q-q^{-1}) [m]_q^2 [2\eta m]_q} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (2.30)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp k}] = \mp \delta_{m+n,0} q^{\pm(\eta+j+k)m} \frac{[cm]_q [\eta m]_q}{m(q-q^{-1}) [m]_q^2 [2\eta m]_q} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (2.31)$$

Here

$$\text{sgn}(l-j) = \begin{cases} + & (l > j), \\ - & (l < j). \end{cases}$$

Then one can realize the simple root type $\alpha_{j,m}$ in terms of the orthonormal basis type $\mathcal{E}_m^{\pm j}$ as follows.

Proposition 2.3.

$$\alpha_{j,m} = \pm [m]_q^2 (q - q^{-1}) (\mathcal{E}_m^{\pm j} - q^{\mp m} \mathcal{E}_m^{\pm(j+1)}), \quad (2.32)$$

$$\alpha_{N,m} = [m]_q (q^{m/2} - q^{-m/2}) (q^{-m/2} \mathcal{E}_m^{+N} - q^{m/2} \mathcal{E}_m^{-N}). \quad (2.33)$$

The following formulae are also useful.

Proposition 2.4. For $1 \leq i, j \leq N$, the following commutation relations hold.

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm j}] = \pm \delta_{m+n,0} \frac{[cm]_q}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} q^{-cm} (q^{\mp m} \delta_{i,j} - \delta_{i,j-1}), \quad (2.34)$$

$$[\mathcal{E}_m^{\pm i}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} e_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}), \quad (2.35)$$

$$[\mathcal{E}_m^{\pm i}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}). \quad (2.36)$$

2.3 The elliptic currents $k_{\pm j}(z)$

Let us set

$$\psi_j(z) =: \exp \left\{ (q - q^{-1}) \sum_{m \neq 0} \frac{\alpha_{j,m}}{1-p^m} p^m z^{-m} \right\} :. \quad (2.37)$$

Then the elliptic currents $\psi_j^\pm(z)$ in Definition 2.1 can be written as

$$\psi_j^+(q^{-\frac{c}{2}}z) = K_j^+ \psi_j(z), \quad \psi_j^-(q^{-\frac{c}{2}}z) = K_j^- \psi_j(pq^{-c}z). \quad (2.38)$$

Let us introduce the new elliptic currents $k_{\pm j}(z)$ ($j \in \{0\} \cup I$) associated with $\mathcal{E}_m^{\pm j}$ by

$$k_{\pm j}(z) = : \exp \left\{ \sum_{m \neq 0} \frac{[m]_q^2 (q - q^{-1})^2}{1 - p^m} p^m \mathcal{E}_m^{\pm j} z^{-m} \right\} :, \quad (2.39)$$

$$k_0(z) = : k_{-N}(q^{-1/2}z) \psi_N(q^{-1/2}z) :=: k_{+N}(q^{1/2}z) \psi_N(q^{1/2}z)^{-1} :. \quad (2.40)$$

Then from Proposition 2.3 we have the following decompositions.

Proposition 2.5.

$$\psi_j(z) = : k_{+j}(z) k_{+(j+1)}(qz)^{-1} :=: k_{-j}(z)^{-1} k_{-(j+1)}(q^{-1}z) :, \quad (2.41)$$

$$\psi_N(z) = : k_{+N}(z) k_0(q^{-1/2}z)^{-1} :=: k_{-N}(z)^{-1} k_0(q^{1/2}z) :. \quad (2.42)$$

In addition, from Proposition 2.2 we obtain the following commutation relations.

Theorem 2.6.

$$\begin{aligned} k_{\pm j}(z_1) k_{\pm j}(z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} k_{\pm j}(z_2) k_{\pm j}(z_1), \quad (1 \leq j \leq N), \\ k_{+j}(q^j z_1) k_{+k}(q^k z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} k_{+k}(q^k z_2) k_{+j}(q^j z_1) \quad (1 \leq j < k \leq N), \\ k_{-j}(q^{-j} z_1) k_{-k}(q^{-k} z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} k_{-k}(q^{-k} z_2) k_{-j}(q^{-j} z_1) \quad (1 \leq k < j \leq N), \\ k_{+j}(q^j z_1) k_{-k}(q^{-k} \xi z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} k_{-k}(q^{-k} \xi z_2) k_{+j}(q^j z_1) \quad (j \neq k), \\ k_{+j}(q^j z_1) k_{-j}(q^{-j} \xi z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{2j-2} \xi^{-1} z) \Theta_p(q^{2j} \xi^{-1} z)}{\Theta_{p^*}(q^{2j} \xi^{-1} z) \Theta_p(q^{2j-2} \xi^{-1} z)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} k_{-j}(q^{-j} \xi z_2) k_{+j}(q^j z_1), \\ k_0(z_1) k_0(z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(q^2 z) \Theta_{p^*}(qz) \Theta_p(q^{-1}z)}{\Theta_{p^*}(q^2 z) \Theta_p(q^{-2}z) \Theta_{p^*}(q^{-1}z) \Theta_p(qz)} k_0(z_2) k_0(z_1), \\ k_{+j}(q^j z_1) k_0(q^{N-1/2} z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} k_0(q^{N-1/2} z_2) k_{+j}(q^j z_1) \quad (1 \leq j \leq N), \\ k_{-j}(\xi q^{-j} z_1) k_0(q^{N-1/2} z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(z) \Theta_p(q^2 z)}{\Theta_{p^*}(q^2 z) \Theta_p(z)} k_0(q^{N-1/2} z_2) k_{-j}(\xi q^{-j} z_1) \quad (1 \leq j \leq N), \end{aligned}$$

where $z = z_1/z_2$, and $\tilde{\rho}^+(z)$ is a function which appears associated with the elliptic dynamical R -matrices [21]. (See §3.1)

$$\tilde{\rho}^+(z) = \frac{\{\xi z\}^2 \{\xi^2 q^{-2} z\} \{q^2 z\}}{\{\xi^2 z\} \{z\} \{\xi q^2 z\} \{\xi q^{-2} z\}} \frac{\{p \xi^2 / z\} \{p / z\} \{p \xi q^2 / z\} \{p \xi q^{-2} / z\}}{\{p \xi / z\}^2 \{p \xi^2 q^{-2} / z\} \{p q^2 / z\}} \quad (2.43)$$

where $\xi = q^{-2\eta}$, $\{z\} = (z; p, \xi^2)_\infty$. We also set $\tilde{\rho}^{+*}(z) = \tilde{\rho}^+(z)|_{p \mapsto p^*}$.

Proposition 2.7.

$$\begin{aligned}
k_{\pm j}(z_1)e_j(z_2) &= \frac{\Theta_{p^*}(q^{-c}z)}{\Theta_{p^*}(q^{-c\mp 2}z)}e_j(z_2)k_{\pm j}(z_1) \quad (1 \leq j \leq N), \\
k_{\pm j}(z_1)e_{j-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{j-1}(z_2)k_{\pm j}(z_1) \quad (2 \leq j \leq N), \\
k_{\pm j}(z_1)e_k(z_2) &= e_k(z_2)k_{\pm j}(z_1) \quad (k \neq j, j-1), \\
k_{\pm j}(z_1)f_j(z_2) &= \frac{\Theta_p(q^{\mp 2}z)}{\Theta_p(z)}f_j(z_2)k_{\pm j}(z_1) \quad (1 \leq j \leq N), \\
k_{\pm j}(z_1)f_{j-1}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{j-1}(z_2)k_{\pm j}(z_1) \quad (2 \leq j \leq N), \\
k_{\pm j}(z_1)f_k(z_2) &= f_k(z_2)k_{\pm j}(z_1) \quad (k \neq j, j-1), \\
k_0(q^{N-1/2}z_1)e_N(z_2) &= \frac{\Theta_{p^*}(q^{-c+N}z)\Theta_{p^*}(q^{-c+N-1}z)}{\Theta_{p^*}(q^{-c+N-2}z)\Theta_p(q^{-c+N+1}z)}e_N(z_2)k_0(q^{N-1/2}z_1), \\
k_0(q^{N-1/2}z_1)e_j(z_2) &= e_j(z_2)k_0(q^{N-1/2}z_1) \quad (1 \leq j \leq N-1), \\
k_0(q^{N-1/2}z_1)f_N(z_2) &= \frac{\Theta_p(q^{N-2}z)\Theta_p(q^{N+1}z)}{\Theta_p(q^Nz)\Theta_p(q^{N-1}z)}f_N(z_2)k_0(q^{N-1/2}z_1), \\
k_0(q^{N-1/2}z_1)f_j(z_2) &= f_j(z_2)k_0(q^{N-1/2}z_1) \quad (1 \leq j \leq N-1).
\end{aligned}$$

2.4 The H -algebra $U_{q,p}(B_N^{(1)})$

Let \mathcal{A} be a complex associative algebra, \mathcal{H} be a finite dimensional commutative subalgebra of \mathcal{A} , and $\mathcal{M}_{\mathcal{H}^*}$ be the field of meromorphic functions on \mathcal{H}^* the dual space of \mathcal{H} .

Definition 2.8 (\mathcal{H} -algebra [5]). *An \mathcal{H} -algebra is an associative algebra \mathcal{A} with 1, which is bigraded over \mathcal{H}^* , $\mathcal{A} = \bigoplus_{\alpha, \beta \in \mathcal{H}^*} \mathcal{A}_{\alpha, \beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : \mathcal{M}_{\mathcal{H}^*} \rightarrow \mathcal{A}_{0,0}$ (the left and right moment maps), such that*

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha \widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta \widehat{f}), \quad a \in \mathcal{A}_{\alpha, \beta}, \quad \widehat{f} \in \mathcal{M}_{\mathcal{H}^*},$$

where T_α denotes the automorphism $(T_\alpha \widehat{f})(\lambda) = \widehat{f}(\lambda + \alpha)$ of $\mathcal{M}_{\mathcal{H}^*}$.

Proposition 2.9. $U = U_{q,p}(B_N^{(1)})$ is a H -algebra by

$$\begin{aligned}
U &= \bigoplus_{\alpha, \beta \in H^*} U_{\alpha, \beta} \\
U_{\alpha, \beta} &= \left\{ x \in U \mid q^{P+h} x q^{-(P+h)} = q^{<\alpha, P+h>} x, \quad q^P x q^{-P} = q^{<\beta, P>} x \quad \forall P+h, P \in H \right\}
\end{aligned}$$

and $\mu_l, \mu_r : \mathbb{F} \rightarrow U_{0,0}$ defined by

$$\mu_l(\widehat{f}) = \widehat{f}(P+h, p) \in \mathbb{F}[[p]], \quad \mu_r(\widehat{f}) = \widehat{f}(P, p^*) \in \mathbb{F}[[p]].$$

We regard $T_\alpha = e^\alpha \in \mathbb{C}[\mathcal{R}_Q]$ as the shift operator $\mathcal{M}_{H^*} \rightarrow \mathcal{M}_{H^*}$

$$(T_\alpha \hat{f}) = e^\alpha \hat{f}(P, p^*) e^{-\alpha} = \hat{f}(P + \langle \alpha, P \rangle, p^*).$$

Hereafter we abbreviate $f(P + h, p)$ and $f(P, p^*)$ as $f(P + h)$ and $f^*(P)$, respectively.

We also consider the H -algebra of the shift operators [5]

$$\begin{aligned} \mathcal{D} &= \left\{ \sum_{\alpha} \hat{f}_{\alpha} T_{\alpha} \mid \hat{f}_{\alpha} \in M_{H^*}, \alpha \in \mathcal{R}_Q \right\}, \\ \mathcal{D}_{\alpha, \alpha} &= \{ \hat{f} T_{-\alpha} \}, \quad \mathcal{D}_{\alpha, \beta} = 0 \quad (\alpha \neq \beta), \\ \mu_l^{\mathcal{D}}(\hat{f}) &= \mu_r^{\mathcal{D}}(\hat{f}) = \hat{f} T_0 \quad \hat{f} \in M_{H^*}. \end{aligned}$$

Then we have the H -algebra isomorphism

$$U \cong U \tilde{\otimes} \mathcal{D} \cong \mathcal{D} \tilde{\otimes} U. \quad (2.44)$$

3 The L -operators and The Dynamical RLL -relations

We introduce the elliptic dynamical R -matrix of the $B_N^{(1)}$ type as a certain gauge transformation of Jimbo-Miwa-Okado's face type Boltzmann weight given in [10]. Then we propose a construction of the L -operator satisfying the RLL -relation by means of the elliptic currents of $U_{q,p}(B_N^{(1)})$.

Hereafter we regard q, p as a generic complex number satisfying $|q|, |p| < 1$ and set $p = q^{2r}$. We also use the following theta functions.

$$[u] = q^{\frac{u^2}{r} - u} \Theta_p(z), \quad [u]^* = q^{\frac{u^2}{r^*} - u} \Theta_{p^*}(z), \quad (3.1)$$

where we set $z = q^{2u}$.

3.1 The elliptic dynamical R -matrix of the $B_N^{(1)}$ type

Let $\mathcal{I} = \{0, \pm 1, \pm 2, \dots, \pm N\}$. We fix the order $1 \prec 2 \prec \dots \prec N \prec 0 \prec -N \prec \dots \prec -2 \prec -1$.

Let us consider the elliptic dynamical R -matrix of the $B_N^{(1)}$ type given by

$$\begin{aligned}
R^+(u, s) &= \rho^+(u) \bar{R}^+(u, s), \\
\bar{R}^+(u, s) &= \left\{ \sum_{\substack{j=1 \\ j \neq 0}}^{-1} E_{j,j} \otimes E_{j,j} + \sum_{\substack{1 \preceq j_1 \prec j_2 \preceq -1 \\ j_2 \neq -j_1}} (b(u, s_{j_1 j_2}) E_{j_1, j_1} \otimes E_{j_2, j_2} + \bar{b}(u) E_{j_2, j_2} \otimes E_{j_1, j_1}) \right. \\
&\quad + \sum_{\substack{1 \preceq j_1 \prec j_2 \preceq -1 \\ j_2 \neq -j_1}} (c(u, s_{j_1 j_2}) E_{j_1, j_2} \otimes E_{j_2, j_1} + \bar{c}(u, s_{j_1 j_2}) E_{j_2, j_1} \otimes E_{j_1, j_2}) \\
&\quad + \sum_{1 \preceq j_1 \prec j_2 \preceq -1} (d(u, s_{j_1}, s_{j_2}) E_{-j_2, j_1} \otimes E_{j_2, -j_1} + \bar{d}(u, s_{j_1}, s_{j_2}) E_{-j_1, j_2} \otimes E_{j_1, -j_2}) \\
&\quad \left. + \sum_{j \in \{1, 2, \dots, -2, -1\}} e_j(u, s) E_{-j, j} \otimes E_{j, -j} \right\}, \tag{3.3}
\end{aligned}$$

where $s = P, P + h$, we set $s_{\pm j} \equiv \pm s_{\epsilon_j}$ for $1 \preceq j \preceq N$, $s_{ij} = s_i - s_j$, $s_0 = -\frac{1}{2}$, and

$$\begin{aligned}
\rho^+(u) &= \xi^{-\frac{1}{r}} C(u, \xi)^{\frac{1}{2}} \rho_0^+(u), \\
\rho_0^+(u) &= q^{-1} z^{\frac{1}{r}} \tilde{\rho}^+(u), \quad C(u, \xi) = \frac{\Theta_{\xi^2}(z)^2 \Theta_{\xi^2}(\xi q^2 z) \Theta_{\xi^2}(\xi q^{-2} z)}{\Theta_{\xi^2}(\xi z)^2 \Theta_{\xi^2}(q^2 z) \Theta_{\xi^2}(q^{-2} z)}, \\
b(u, s) &= \frac{[s+1][s-1][u]}{[s]^2[u+1]}, \quad \bar{b}(u) = \frac{[u]}{[u+1]}, \\
c(u, s) &= \frac{[1][s+u]}{[s][u+1]}, \quad \bar{c}(u, s) = \frac{[1][s-u]}{[s][u+1]},
\end{aligned}$$

$$\begin{aligned}
d(u, s_j, s_k) &= G_{s_j} \frac{[u][1][s_j + s_k + 1 + \eta - u]}{[\eta - u][u+1][s_j + s_k + 1]} \prod_{m=1}^{j-1} \frac{[s_j - s_m]}{[s_j - s_m + 1]} \prod_{m=1}^{k-1} \frac{[s_k - s_m + 1]}{[s_k - s_m]} \\
&\quad (j \prec k \preceq 0) \\
d(u, s_{-k}, s_{-j}) &= G_{s_{-j}} \frac{[u][1][s_{-j} + s_{-k} + 1 + \eta - u]}{[\eta - u][u+1][s_{-j} + s_{-k} + 1]} \prod_{m=1}^{j-1} \frac{[s_{-j} + s_m]}{[s_{-j} + s_m + 1]} \prod_{m=1}^{k-1} \frac{[s_{-k} + s_m + 1]}{[s_{-k} + s_m]} \\
&\quad (0 \preceq -k \prec -j) \\
d(u, s_j, s_{-k}) &= G_{s_j} G_{s_{-k}} \frac{[u][1][s_j + s_{-k} + 1 + \eta - u]}{[\eta - u][u+1][s_j + s_{-k} + 1]} \prod_{m=1}^{j-1} \frac{[s_j - s_m]}{[s_j - s_m + 1]} \prod_{m=1}^{k-1} \frac{[s_{-k} + s_m]}{[s_{-k} + s_m + 1]} \\
&\quad (j \prec 0 \prec -k),
\end{aligned}$$

$$\begin{aligned}
\bar{d}(u, s_j, s_k) &= G_{s_k} \frac{[u][1][s_j + s_k + 1 + \eta - u]}{[\eta - u][u + 1][s_j + s_k + 1]} \prod_{m=1}^{j-1} \frac{[s_j - s_m + 1]}{[s_j - s_m]} \prod_{m=1}^{k-1} \frac{[s_k - s_m]}{[s_k - s_m + 1]} \\
&\quad (j \prec k \preceq 0), \\
\bar{d}(u, s_{-k}, s_{-j}) &= G_{s_{-k}} \frac{[u][1][s_{-j} + s_{-k} + 1 + \eta - u]}{[\eta - u][u + 1][s_{-j} + s_{-k} + 1]} \prod_{m=1}^{j-1} \frac{[s_{-j} + s_m + 1]}{[s_{-j} + s_m]} \prod_{m=1}^{k-1} \frac{[s_{-k} + s_m]}{[s_{-k} + s_m + 1]} \\
&\quad (0 \preceq -k \prec -j), \\
\bar{d}(u, s_j, s_{-k}) &= \frac{[u][1][s_j + s_{-k} + 1 + \eta - u]}{[\eta - u][u + 1][s_j + s_{-k} + 1]} \prod_{m=1}^{j-1} \frac{[s_j - s_m + 1]}{[s_j - s_m]} \prod_{m=1}^{k-1} \frac{[s_{-k} + s_m + 1]}{[s_{-k} + s_m]} \\
&\quad (j \preceq 0 \preceq -k),
\end{aligned}$$

$$\begin{aligned}
e_j(u, s) &= \frac{[1][2s_j + 1 - u]}{[u + 1][2s_j + 1]} + \frac{[u][1][2s_j + 1 + \eta - u]}{[\eta - u][u + 1][2s_j + 1]} G_{s_j} \quad (j \neq 0), \\
e_0(u, s) &= \frac{[\eta + u][1][2\eta - u]}{[\eta - u][u + 1][2\eta]} - \frac{[u][1]}{[u + 1][2\eta]} H_s.
\end{aligned}$$

where for $k, -k = 0$ the product $\prod_{m=1}^{k-1}$ should be understood as $\prod_{m=1}^N$ etc. We also set

$$\begin{aligned}
G_{s_j} &= \frac{[s_j + 1]}{[s_j]} \prod_{\substack{k \in \mathcal{I} \\ \neq \pm j, 0}} \frac{[s_j - s_k + 1]}{[s_j - s_k]} = \frac{[s_j + 1]}{[s_j]} \prod_{\substack{m=1 \\ \neq |j|}}^N \frac{[s_j - s_m + 1]}{[s_j - s_m]} \prod_{\substack{m=1 \\ \neq |j|}}^N \frac{[s_j + s_m + 1]}{[s_j + s_m]}, \\
H_s &= \sum_{\substack{k \in \mathcal{I} \\ \neq 0}} \frac{[s_k + \frac{1}{2} + 2\eta]}{[s_k + \frac{1}{2}]} G_{s_k}.
\end{aligned}$$

Note that

$$G_{s_{j-1}} = G_{s_{-j}}^{-1}, \quad G_{s_{-j-1}} = G_{s_j}^{-1},$$

and

$$\rho^+(u)\rho^+(-u) = 1, \quad \rho^+(\eta - u) = \rho^+(u) \frac{[u][\eta - u + 1]}{[u + 1][\eta - u]}. \quad (3.4)$$

The matrix $R^+(u, s)$ in (3.3) is related to Jimbo-Miwa-Okado's face type Botzmann weight [10] by the gauge transformation. For $j \in \mathcal{I}$, we set $\hat{j} = \epsilon_j$ for $1 \preceq j \preceq N$, $\hat{j} = -\epsilon_{|j|}$ for $-N \preceq j \preceq -1$ and $\hat{0} = 0$. Let us define $F(s, s + \hat{j}) = \left(\frac{G_{s_j}}{G_{s_j}(j)} \right)^{\frac{1}{2}}$ with

$$G_{s_j}(j) = \begin{cases} \prod_{m=1}^{j-1} \frac{[s_j - s_m + 1]}{[s_j - s_m]} & \text{if } j \prec 0 \\ \frac{[s_j + 1]}{[s_j]} \prod_{\substack{m=1 \\ \neq |j|}}^N \frac{[s_j - s_m + 1]}{[s_j - s_m]} \prod_{m=|j|+1}^N \frac{[s_j + s_m + 1]}{[s_j + s_m]} & \text{if } 0 \prec j \end{cases}$$

For $a \in \widetilde{\mathfrak{h}}^*$, $\rho = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_N$ and $a_j = \langle a + \rho, \widehat{j} \rangle$, we identify a_j with s_j . Then we have

$$\begin{aligned}
R^+(u, s)_{kl}^{ij} &= W \left(\begin{array}{cc|c} a & a + \widehat{i} & u \\ a + \widehat{l} & a + \widehat{i} + \widehat{j} & \end{array} \right) \quad (i + j = k + l), \\
W \left(\begin{array}{cc|c} a & a + \widehat{i} & u \\ a + \widehat{l} & a + \widehat{i} + \widehat{j} & \end{array} \right) &= \rho^+(u) \frac{[\eta][1]}{[\eta - u][u + 1]} \frac{F(a, a + \widehat{i})F(a + \widehat{i}, a + \widehat{i} + \widehat{j})}{F(a, a + \widehat{l})F(a + \widehat{l}, a + \widehat{l} + \widehat{k})} \\
&\quad \times W_{JMO} \left(\begin{array}{cc|c} a & a + \widehat{i} & u \\ a + \widehat{l} & a + \widehat{i} + \widehat{j} & \end{array} \right) \quad (3.5)
\end{aligned}$$

One can derive the following relations from (2.10)-(2.13b) in [10].

1) Crossing symmetry

$$W \left(\begin{array}{cc|c} a & b & u \\ d & c & \end{array} \right) = \frac{F(b, c)F(c, b)}{F(a, d)F(d, a)} \left(\frac{G_b G_d}{G_a G_c} \right)^{\frac{1}{2}} W \left(\begin{array}{cc|c} d & a & \eta - u \\ c & b & \end{array} \right). \quad (3.6)$$

2) Reflection symmetry

$$W \left(\begin{array}{cc|c} a & b & u \\ d & c & \end{array} \right) = \left(\frac{F(a, b)F(b, c)}{F(a, d)F(d, c)} \right)^2 W \left(\begin{array}{cc|c} a & d & u \\ b & c & \end{array} \right).$$

3) Unitarity

$$\sum_g W \left(\begin{array}{cc|c} a & g & u \\ d & c & \end{array} \right) W \left(\begin{array}{cc|c} a & b & -u \\ g & c & \end{array} \right) = \delta_{bd}.$$

4) 2nd inversion relation

$$\sum_g \left(\frac{G_a G_g}{G_b G_d} \right)^{\frac{1}{2}} W \left(\begin{array}{cc|c} a & b & \eta - u \\ d & g & \end{array} \right) \left(\frac{G_c G_g}{G_b G_d} \right)^{\frac{1}{2}} W \left(\begin{array}{cc|c} c & d & \eta + u \\ b & g & \end{array} \right) = \delta_{ac}.$$

Here

$$G_a = \varepsilon(a) \prod_{j=1}^N [a_j] \prod_{1 \leq i < j \leq N} [a_i - a_j][a_i + a_j] \quad (3.7)$$

and $\varepsilon(a)$ is a sign factor such that $\varepsilon(a + \widehat{j})/\varepsilon(a) = 1$.

Remark. The choice of the gauge (3.5) and the resultant R -matrix (3.3) is convenient to discuss the RLL -relations in the next sections, because it allows the L -operator $\widehat{L}^-(u)$ to be related to $\widehat{L}^+(u)$ simply by $\widehat{L}^-(u) = \widehat{L}^+(u + r - \frac{c}{2})$, i.e. one needs no extra modifications follow from

Proposition 4.3 in [12] (4.8). Note that p in [12] is p^* in the present paper. See also [13]. One drawback is that one needs to introduce a set of extra generators and a central extension to the group algebra $\mathbb{C}[\mathcal{R}_Q]$ in order to remove constant gauge factors such as $q^{1/r}$ and q^{1/r^*} in a realization of the proper modified elliptic currents, which will be discussed in the next section. See Remark below Proposition 3.10 in [18]. However in order to avoid such unessential complications, we hereafter treat the whole formulas up to those constant gauge factors.

3.2 Modified elliptic currents

Since our elliptic R -matrix is given by the theta functions (3.1) accompanied by the fractional power of z , we need to introduce the following modifications of the elliptic currents.

$$\begin{aligned} E_j(u) &= e_j(z) z^{-\frac{P\alpha_j-1}{r^*}} \quad (1 \leq j \leq N-1), \\ E_N(u) &= e_N(z) z^{-\frac{P\alpha_N-1/2}{r^*}}, \\ F_j(u) &= f_j(z) z^{\frac{P\alpha_j+h\alpha_j-1}{r}} \quad (1 \leq j \leq N-1), \\ F_N(u) &= f_N(z) z^{\frac{P\alpha_N+h\alpha_N-1/2}{r}}, \\ H_j^\pm(z) &= \psi_j^\pm(z) (K_j^\pm)^{-1} e^{-Q_{\alpha_j}} (q^{\pm(r-\frac{c}{2})} z)^{-\frac{r-r^*}{rr^*}(P\alpha_j-1)+\frac{1}{r}h\alpha_j}, \end{aligned}$$

and

$$\begin{aligned} K_{+j}^+(u) &= k_{+j}(q^j z) e^{-Q_{\epsilon_j}} (q^j z q^{-r})^{-\frac{r-r^*}{rr^*}P_{\epsilon_j}+\frac{1}{r}h_{\epsilon_j}}, \\ K_{-j}^+(u) &= k_{-j}(q^{-j} \xi z) e^{Q_{\epsilon_j}} (q^{-j} \xi z q^{-r})^{\frac{r-r^*}{rr^*}P_{\epsilon_j}-\frac{1}{r}h_{\epsilon_j}}, \\ K_0^+(u) &= k_0(\xi^{1/2} z), \\ K_{\pm j}^-(u) &= K_{\pm j}^+ \left(u + r - \frac{c}{2} \right). \end{aligned}$$

for $1 \leq j \leq N$. We also set

$$\tilde{d} = \hat{d} + \frac{1}{2r^*} \sum_{j=1}^N (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^N ((P + h)_j + 2) (P + h)^j.$$

We have

Proposition 3.1.

$$\begin{aligned} H_j^\pm(u) &= : K_{+j}^\pm \left(u + \frac{c}{4} - \frac{j}{2} \right) K_{+(j+1)}^\pm \left(u + \frac{c}{4} - \frac{j}{2} \right)^{-1} :, \\ &= : K_{-j}^\pm \left(u + \frac{c}{4} + \frac{j}{2} + \eta \right)^{-1} K_{-(j+1)}^\pm \left(u + \frac{c}{4} + \frac{j}{2} + \eta \right) : \quad (1 \leq j \leq N-1), \\ H_N^\pm(u) &= : K_{+N}^\pm \left(u + \frac{c}{4} - \frac{N}{2} \right) K_0^\pm \left(u + \frac{c}{4} - \frac{N}{2} \right)^{-1} :, \\ &= : K_{-N}^\pm \left(u + \frac{c}{4} + \frac{N}{2} + \eta \right)^{-1} K_0^\pm \left(u + \frac{c}{4} + \frac{N}{2} + \eta \right) : . \end{aligned}$$

Then one can rewrite the formulas in Theorem 2.6 and Proposition 2.7 as follows.

Proposition 3.2.

$$\begin{aligned}
K_{\pm j}^+(u_1)K_{\pm j}^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} K_{\pm j}^+(u_2)K_{\pm j}^+(u_1), \\
K_{+j}^+(u_1)K_{+l}^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} \frac{[u_1 - u_2 - 1]^*[u_1 - u_2]}{[u_1 - u_2]^*[u_1 - u_2 - 1]} K_{+l}^+(u_2)K_{+j}^+(u_1) \quad (1 \preceq j \prec l \preceq 0), \\
K_{-j}^+(u_1)K_{-l}^+(u_2) &= \frac{\rho^+(u_1 - u_2)}{\rho^{+*}(u_1 - u_2)} \frac{[u_1 - u_2]^*[u_1 - u_2 - 1]}{[u_1 - u_2 - 1]^*[u_1 - u_2]} K_{-l}^+(u_2)K_{-j}^+(u_1) \quad (1 \preceq j \prec l \preceq 0), \\
K_{+j}^+(u_1)K_{-j}^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} \frac{[u_1 - u_2 + \eta + j - 1]^*[u_1 - u_2 - 1]^*}{[u_1 - u_2 + \eta + j]^*[u_1 - u_2]^*} \\
&\quad \times \frac{[u_1 - u_2 + \eta + j][u_1 - u_2]}{[u_1 - u_2 + \eta + j - 1][u_1 - u_2 - 1]} K_{-l}^+(u_2)K_{+j}^+(u_1), \\
K_{+j}^+(u_1)K_{-l}^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} \frac{[u_1 - u_2 - 1]^*[u_1 - u_2]}{[u_1 - u_2]^*[u_1 - u_2 - 1]} K_{-l}^+(u_2)K_{+j}^+(u_1) \quad (1 \preceq j, l \preceq N, j \neq l), \\
K_0^+(u_1)K_0^+(u_2) &= \frac{\rho^{+*}(u_1 - u_2)}{\rho^+(u_1 - u_2)} \frac{[u_1 - u_2 - 1]^*[u_1 - u_2 + \frac{1}{2}]^*}{[u_1 - u_2 + 1]^*[u_1 - u_2 - \frac{1}{2}]^*} \\
&\quad \times \frac{[u_1 - u_2 + 1][u_1 - u_2 - \frac{1}{2}]}{[u_1 - u_2 - 1][u_1 - u_2 + \frac{1}{2}]} K_0^+(u_2)K_0^+(u_1).
\end{aligned}$$

Proposition 3.3.

$$\begin{aligned}
K_{+j}^+(u_1)E_j(u_2) &= \frac{[u_1 - u_2 + \frac{j-c}{2}]^*}{[u_1 - u_2 + \frac{j-c}{2} - 1]^*} E_j(u_2)K_{+j}^+(u_1) \quad (1 \leq j \leq N), \\
K_{+j}^+(u_1)E_{j-1}(u_2) &= \frac{[u_1 - u_2 + \frac{j-1-c}{2}]^*}{[u_1 - u_2 + \frac{j-1-c}{2} + 1]^*} E_{j-1}(u_2)K_{+j}^+(u_1) \quad (2 \leq j \leq N), \\
K_j^+(u_1)E_l(u_2) &= E_l(u_2)K_j^+(u_1) \quad (l \neq j, j-1), \\
K_{-j}^+(u_1)E_j(u_2) &= \frac{[u_1 - u_2 - \frac{j+c}{2} - \eta]^*}{[u_1 - u_2 - \frac{j+c}{2} - \eta + 1]^*} E_j(u_2)K_{-j}^+(u_1) \quad (1 \leq j \leq N), \\
K_{-j}^+(u_1)E_{j-1}(u_2) &= \frac{[u_1 - u_2 - \frac{j-1+c}{2} - \eta]^*}{[u_1 - u_2 - \frac{j-1+c}{2} - \eta - 1]^*} E_{j-1}(u_2)K_{-j}^+(u_1) \quad (2 \leq j \leq N), \\
K_{-j}^+(u_1)E_l(u_2) &= E_l(u_2)K_{-j}^+(u_1) \quad (l \neq j, j-1),
\end{aligned}$$

$$\begin{aligned}
K_{+j}^+(u_1)F_j(u_2) &= \frac{\left[u_1 - u_2 + \frac{j}{2} - 1\right]}{\left[u_1 - u_2 + \frac{j}{2}\right]} F_j(u_2)K_{+j}^+(u_1) \quad (1 \leq j \leq N), \\
K_{+j}^+(u_1)F_{j-1}(u_2) &= \frac{\left[u_1 - u_2 + \frac{j+1}{2}\right]}{\left[u_1 - u_2 + \frac{j+1}{2} - 1\right]} F_{j-1}(u_2)K_{+j}^+(u_1) \quad (2 \leq j \leq N), \\
K_j^+(u_1)F_l(u_2) &= F_l(u_2)K_j^+(u_1) \quad (l \neq j, j-1), \\
K_{-j}^+(u_1)F_j(u_2) &= \frac{\left[u_1 - u_2 - \frac{j}{2} - \eta + 1\right]}{\left[u_1 - u_2 - \frac{j}{2} - \eta\right]} F_j(u_2)K_{-j}^+(u_1) \quad (1 \leq j \leq N), \\
K_{-j}^+(u_1)F_{j-1}(u_2) &= \frac{\left[u_1 - u_2 - \frac{j+1}{2} - \eta\right]}{\left[u_1 - u_2 - \frac{j+1}{2} - \eta + 1\right]} F_{j-1}(u_2)K_{-j}^+(u_1) \quad (2 \leq j \leq N), \\
K_{-j}^+(u_1)F_l(u_2) &= F_l(u_2)K_{-j}^+(u_1) \quad (l \neq j, j-1),
\end{aligned}$$

and

$$\begin{aligned}
K_0^+(u_1)E_N(u_2) &= \frac{\left[u_1 - u_2 + \frac{N-c}{2}\right]^*}{\left[u_1 - u_2 + \frac{N-c}{2} - 1\right]^*} \frac{\left[u_1 - u_2 + \frac{N-c-1}{2}\right]^*}{\left[u_1 - u_2 + \frac{N-c+1}{2}\right]^*} E_N(u_2)K_0^+(u_1), \\
K_0^+(u_1)E_j(u_2) &= E_j(u_2)K_0^+(u_1) \quad (j \neq N, 0), \\
K_0^+(u_1)F_N(u_2) &= \frac{\left[u_1 - u_2 + \frac{N}{2} - 1\right]}{\left[u_1 - u_2 + \frac{N}{2}\right]} \frac{\left[u_1 - u_2 + \frac{N+1}{2}\right]}{\left[u_1 - u_2 + \frac{N-1}{2}\right]} F_N(u_2)K_0^+(u_1), \\
K_0^+(u_1)F_j(u_2) &= F_j(u_2)K_0^+(u_1) \quad (j \neq N, 0).
\end{aligned}$$

In addition, the defining relations (2.5)–(2.20) of $U_{q,p}(B_N^{(1)})$ can be rewritten as follows in the sense of analytic continuation.

Proposition 3.4.

$$[h_i, \alpha_{j,n}] = 0, \quad [h_i, E_j(u)] = a_{ij}E_j(u), \quad [h_i, F_j(u)] = -a_{ij}F_j(u), \quad (3.8)$$

$$[\widehat{d}, h_i] = 0, \quad [\widehat{d}, \alpha_{i,n}] = n\alpha_{i,n}, \quad (3.9)$$

$$[\widehat{d}, E_i(u)] = \left(-z\frac{\partial}{\partial z} + \frac{1}{r^*}\right)E_i(u), \quad [\widehat{d}, F_i(u)] = \left(-z\frac{\partial}{\partial z} + \frac{1}{r}\right)F_i(u), \quad (3.10)$$

$$\left[u - v - \frac{b_{ij}}{2}\right]^* E_i(u)E_j(v) = \left[u - v + \frac{b_{ij}}{2}\right]^* E_j(v)E_i(u), \quad (3.11)$$

$$\left[u - v + \frac{b_{ij}}{2}\right] F_i(u)F_j(v) = \left[u - v - \frac{b_{ij}}{2}\right] F_j(v)F_i(u), \quad (3.12)$$

$$[E_i(u), F_j(v)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-c}\frac{z}{w})H_i^-(q^{c/2}w) - \delta(q^c\frac{z}{w})H_i^+(q^{-c/2}w) \right), \quad (3.13)$$

$$\begin{aligned}
& \sum_{\sigma \in S_a} z_{\sigma(1)}^{-\frac{1}{r^*}} \prod_{1 \leq k < m \leq a} \frac{(p^* q^2 z_{\sigma(m)} / z_{\sigma(k)}; p^*)_{\infty}}{(p^* q^{-2} z_{\sigma(m)} / z_{\sigma(k)}; p^*)_{\infty}} \\
& \times \sum_{s=0}^a (-)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{s+1 \leq m \leq a-1} \left(\frac{w}{z_{\sigma(m)}} \right)^{\frac{1}{r^*}} \prod_{a \leq m \leq s} \left(\frac{z_{\sigma(m)}}{w} \right)^{\frac{1}{r^*}} \\
& \times \prod_{1 \leq m \leq s} \frac{(p^* q^{b_{ij}} w / z_{\sigma(m)}; p^*)_{\infty}}{(p^* q^{-b_{ij}} w / z_{\sigma(m)}; p^*)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{b_{ij}} z_{\sigma(m)} / w; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_{\sigma(m)} / w; p^*)_{\infty}} \\
& \times E_i(u_{\sigma(1)}) \cdots E_i(u_{\sigma(s)}) E_j(v) E_i(u_{\sigma(s+1)}) \cdots E_i(u_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}),
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
& \sum_{\sigma \in S_a} z_{\sigma(1)}^{\frac{1}{r}} \prod_{1 \leq k < m \leq a} \frac{(pq^{-2} z_{\sigma(m)} / z_{\sigma(k)}; p)_{\infty}}{(pq^2 z_{\sigma(m)} / z_{\sigma(k)}; p)_{\infty}} \\
& \times \sum_{s=0}^a (-)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{s+1 \leq m \leq a-1} \left(\frac{w}{z_{\sigma(m)}} \right)^{-\frac{1}{r}} \prod_{a \leq m \leq s} \left(\frac{z_{\sigma(m)}}{w} \right)^{-\frac{1}{r}} \\
& \times \prod_{1 \leq m \leq s} \frac{(pq^{-b_{ij}} w / z_{\sigma(m)}; p)_{\infty}}{(pq^{b_{ij}} w / z_{\sigma(m)}; p)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(pq^{-b_{ij}} z_{\sigma(m)} / w; p)_{\infty}}{(pq^{b_{ij}} z_{\sigma(m)} / w; p)_{\infty}} \\
& \times F_i(u_{\sigma(1)}) \cdots F_i(u_{\sigma(s)}) F_j(v) F_i(u_{\sigma(s+1)}) \cdots F_i(u_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}).
\end{aligned} \tag{3.15}$$

3.3 The half currents and the L -operators

We next introduce the half currents $E_{i,j}^{\pm}(u), F_{j,i}^{\pm}(u)$ ($1 \preceq i \prec j \preceq -1$) and propose a construction of the L -operators of $U_{q,p}(B_N^{(1)})$.

Definition 3.5. For $1 \preceq j \preceq -2$, we define the basic half currents $E_{j+1,j}^+(u), F_{j,j+1}^+(u)$ as follows.

$$\begin{aligned}
F_{j,j+1}^+(u) &:= a_{j,j+1} \oint_C \frac{dz'_j}{2\pi i z'_j} F_j(u'_j) \frac{[u - u'_j + (P+h)_{j,j+1} + \frac{j}{2} - 1][1]}{[u - u'_j + \frac{j}{2}][(P+h)_{j,j+1} - 1]}, \\
E_{j+1,j}^+(u) &:= a_{j+1,j}^* \oint_{C^*} \frac{dz'_j}{2\pi i z'_j} E_j(u'_j) \frac{[u - u'_j + \frac{j-c}{2} + 1 - P_{j,j+1}]^*[1]^*}{[u - u'_j + \frac{j-c}{2}]^*[P_{j,j+1} - 1]^*},
\end{aligned}$$

for $1 \preceq j \preceq N$ and

$$\begin{aligned}
F_{-(j+1),-j}^+(u) &:= a_{-(j+1),-j} \oint_C \frac{dz'_j}{2\pi i z'_j} F_j(u'_j) \frac{[u - u'_j + (P+h)_{-(j+1),-j} - \frac{j}{2} - \eta - 1][1]}{[u - u'_j - \frac{j}{2} - \eta][(P+h)_{-(j+1),-j} - 1]}, \\
E_{-j,-(j+1)}^+(u) &:= a_{-j,-(j+1)}^* \oint_{C^*} \frac{dz'_j}{2\pi i z'_j} E_j(u'_j) \frac{[u - u'_j - \frac{j+c}{2} - \eta + 1 - P_{-(j+1),-j} - \delta_{j,N}]^*[1]^*}{[u - u'_j - \frac{j+c}{2} - \eta]^*[P_{-(j+1),-j} - 1 + \delta_{j,N}]^*},
\end{aligned}$$

for $-N \preceq -j \preceq -1$, where $N+1 \equiv 0 \equiv -N-1$. We also define

$$E_{j+1,j}^-(u) := E_{j+1,j}^+(u + r - \frac{c}{2}), \quad F_{j,j+1}^-(u) := F_{j,j+1}^+(u + r - \frac{c}{2}).$$

By using Propositions 3.2-3.4 one can derive the following relations.

Proposition 3.6.

$$\begin{aligned}
K_{j+1}^+(u_1)^{-1} E_{j+1,j}^+(u_2) K_{j+1}^+(u_1) &= E_{j+1,j}^+(u_2) \frac{1}{\bar{b}^*(u)} - E_{j+1,j}^+(u_1) \frac{c^*(u, P_{j,j+1})}{\bar{b}^*(u)}, \\
K_{j+1}^+(u_1) F_{j,j+1}^+(u_2) K_{j+1}^+(u_1)^{-1} &= \frac{1}{\bar{b}(u)} F_{j,j+1}^+(u_2) - \frac{\bar{c}(u, P_{j,j+1} + h_{j,j+1})}{\bar{b}(u)} F_{j,j+1}^+(u_1), \\
\frac{[1-u]^*}{[u]^*} E_{j+1,j}^+(u_1) E_{j+1,j}^+(u_2) + \frac{[1+u]^*}{[u]^*} E_{j+1,j}^+(u_2) E_{j+1,j}^+(u_1) \\
&= E_{j+1,j}^+(u_1)^2 \frac{[1]^* [P_{j,j+1} - 2 + u]^*}{[P_{j,j+1} - 2]^* [u]^*} + E_{j+1,j}^+(u_2)^2 \frac{[1]^* [P_{j,j+1} - 2 - u]^*}{[P_{j,j+1} - 2]^* [u]^*}, \\
\frac{[1+u]}{[u]} F_{j,j+1}^+(u_1) F_{j,j+1}^+(u_2) + \frac{[1-u]}{[u]} F_{j,j+1}^+(u_2) F_{j,j+1}^+(u_1) \\
&= F_{j,j+1}^+(u_1)^2 \frac{[1] [P_{j,j+1} + h_{j,j+1} - 2 - u]}{[P_{j,j+1} + h_{j,j+1} - 2] [u]} + F_{j,j+1}^+(u_2)^2 \frac{[1] [P_{j,j+1} + h_{j,j+1} - 2 + u]}{[P_{j,j+1} + h_{j,j+1} - 2] [u]}, \\
[E_{j+1,j}^+(u_1), F_{j,j+1}^+(u_2)] \\
&= K_j^+(u_2) \frac{\bar{c}^*(u, P_{j,j+1})}{\bar{b}^*(u)} K_{j+1}^+(u_2)^{-1} - K_{j+1}^+(u_1)^{-1} \frac{\bar{c}(u, (P+h)_{j,j+1})}{\bar{b}(u)} K_j^+(u_1).
\end{aligned}$$

Proposition 3.7. For $1 \preceq j \preceq -2, j \neq N, 0$ the relations in Proposition 3.6 and those for $K_j^+(u), K_{j+1}^+(u)$ in Proposition 3.2 coincide with the following RLL-relation of the $U_{q,p}(\widehat{\mathfrak{gl}}_2)$ type.

$$\begin{aligned}
R_{j,j+1}^{+(12)}(u_1 - u_2, P + h) \widehat{L}_{j,j+1}^{+(1)}(u_1) \widehat{L}_{j,j+1}^{+(2)}(u_2) = \\
\widehat{L}_{j,j+1}^{+(2)}(u_2) \widehat{L}_{j,j+1}^{+(1)}(u_1) R_{j,j+1}^{*+(12)}(u_1 - u_2, P - h^{(1)} - h^{(2)}), \tag{3.16}
\end{aligned}$$

where

$$\begin{aligned}
R_{j,j+1}^+(u, s) &= \rho^+(u) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u, s_{j,j+1}) & c(u, s_{j,j+1}) & 0 \\ 0 & \bar{c}(u, s_{j,j+1}) & \bar{b}(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\widehat{L}_{j,j+1}(u) &= \begin{pmatrix} 1 & F_{j,j+1}^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_j^+(u) & 0 \\ 0 & K_{j+1}^+(u) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E_{j+1,j}^+(u) & 1 \end{pmatrix}.
\end{aligned}$$

Definition 3.8. By means of the basic half currents $E_{j+1,j}^\pm(u), F_{j,j+1}^\pm(u)$ ($1 \preceq j \preceq -2$), we define the other half currents $E_{i,j}^\pm(u), F_{j,i}^\pm(u)$ ($1 \preceq i \prec j \preceq -1, j = i + 1$) by requiring the following conditions.

1) The half currents $E_{j,i}^{\pm}(u)$ and $F_{i,j}^{\pm}(u)$ ($i \prec j$) have the following series expansions.

$$E_{j,i}^+(u) = \left(\sum_{n \in \mathbb{Z}_{\geq 0}} E_{j,i;-n}^+ z^n + \sum_{n \in \mathbb{Z}_{> 0}} E_{j,i;n}^+ p^n z^{-n} \right) z^{-\frac{P_{i,j}-1+\delta_{i,0}-\delta_{j,-i}}{r^*}},$$

$$F_{i,j}^+(u) = \left(\sum_{n \in \mathbb{Z}_{\geq 0}} F_{i,j;-n}^+ z^n + \sum_{n \in \mathbb{Z}_{> 0}} F_{i,j;n}^+ p^n z^{-n} \right) z^{\frac{(P+h)_{i,j}-1+\delta_{i,0}-\delta_{j,-i}}{r}},$$

where $E_{j,i;\pm n}^+, F_{i,j;\pm n}^+ \in U_{q,p}(B_N^{(1)})$ and

$$E_{j,i}^-(u) := E_{j,i}^+(u + r - \frac{c}{2}), \quad F_{i,j}^-(u) := F_{i,j}^+(u + r - \frac{c}{2}), \quad K_j^-(u) = K_j^+(u + r - \frac{c}{2}). \quad (3.17)$$

2) Let us set

$$\widehat{L}^+(u) = \begin{pmatrix} 1 & F_{1,2}^+(u) & F_{1,3}^+(u) & \cdots & F_{1,-1}^+(u) \\ 0 & 1 & F_{2,3}^+(u) & \cdots & F_{2,-1}^+(u) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & F_{-2,-1}^+(u) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} K_1^+(u) & 0 & \cdots & 0 \\ 0 & K_2^+(u) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_{-1}^+(u) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ E_{2,1}^+(u) & 1 & \ddots & & \vdots \\ E_{3,1}^+(u) & E_{3,2}^+(u) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ E_{-1,1}^+(u) & E_{-1,2}^+(u) & \cdots & E_{-1,-2}^+(u) & 1 \end{pmatrix}. \quad (3.18)$$

Due to 1), the matrix products in (3.18) are well defined in the p -adic topology. Then $\widehat{L}^+(u)$ and $\widehat{L}^-(u) = \widehat{L}^+(u + r - c/2)$ satisfy the following RLL-relations.

$$R^{\pm(12)}(u, P+h) \widehat{L}^{\pm(1)}(u_1) \widehat{L}^{\pm(2)}(u_2) = \widehat{L}^{\pm(2)}(u_2) \widehat{L}^{\pm(1)}(u_1) R^{*\pm(12)}(u, P-h^{(1)}-h^{(2)}), \quad (3.19)$$

$$R^{\pm(12)}(u \pm \frac{c}{2}, P+h) \widehat{L}^{\pm}(u_1) \widehat{L}^{\mp}(u_2) = \widehat{L}^{\mp}(u_2) \widehat{L}^{\pm}(u_1) R^{*\pm(12)}(u \mp \frac{c}{2}, P-h^{(1)}-h^{(2)}). \quad (3.20)$$

Here

$$R^-(u, s) = R^+(u + r - \frac{c}{2}, s).$$

Note also that $r^* + \frac{c}{2} = r - \frac{c}{2}$.

Remark. Since $\rho^+(u)/\rho^{+*}(u) = \rho_0^+(u)/\rho_0^{+*}(u)$, the *RLL*-relations (3.16), (3.19), (3.20) remain unchanged even if one uses $\rho_0^+(u)$ and $\rho_0^{+*}(u)$ instead of $\rho^+(u)$ and $\rho^{+*}(u)$, respectively. See Sec.6

Conjecture 3.9. *The RLL-relation (3.19) and (3.20) determines the half currents $E_{j,i}^\pm(u), F_{i,j}^\pm(u)$ ($1 \preceq i \prec j \preceq -1, j \neq i+1$) recursively and uniquely from the basic ones in Definition 3.5.*

In fact, the half currents $E_{j,i}^\pm(u), F_{i,j}^\pm(u)$ with $1 \preceq i \prec j \prec 0$ or $0 \prec i \prec j \preceq -1$ are determined recursively by the basic ones in the same way as for $U_{q,p}(A_N^{(1)})$ case [2, 24]. As for the other half currents $E_{j,i}^\pm(u), F_{i,j}^\pm(u)$ with $1 \preceq i \preceq 0 \prec j \preceq -1$ or $1 \preceq i \prec 0 \preceq j \preceq -1$, we have observed that the combinations $E_{j,i}^+(u+c/4) - E_{j,i}^-(u-c/4)$ and $F_{i,j}^+(u-c/4) - F_{i,j}^-(u+c/4)$ satisfy a system of linear equations with the operator valued coefficients given by the *total* elliptic currents respectively. In addition, the half currents $E_{-1,1}^\pm(u)$ (reps. $F_{1,-1}^\pm(u)$) is determined by all the other half currents $E_{j,i}^\pm(u)$ (resp. $F_{i,j}^\pm(u)$) $i \prec j$. An explicit expression for the half currents $E_{j,i}^\pm(u), F_{i,j}^\pm(u)$ ($1 \preceq i \prec j \prec 0$ or $0 \prec i \prec j \preceq -1$) and conjectural expressions for the other half currents are given in Appendix B.

The existence of the operator $\widehat{L}^+(u)$ satisfying (3.19)-(3.20), and hence the existence and the uniqueness of the half currents, can also be seen in the following argument. Consider the elliptic quantum group $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ realized by the Chevalley generators equipped with the quasi-Hopf algebra structure [12]. See Appendix A. Note that $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$ is isomorphic to the Drinfeld-Jimbo's quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ [3, 9] as an associative algebra. In addition, we have shown the isomorphism [6]

$$U_{q,p}(\widehat{\mathfrak{g}})/pU_{q,p}(\widehat{\mathfrak{g}}) \cong (\mathcal{M}_{H^*} \otimes_{\mathbb{C}} U_q(\widehat{\mathfrak{g}})) \# \mathbb{C}[\mathcal{R}_Q]$$

where $U_q(\widehat{\mathfrak{g}})$ is the quantum affine algebra in the Drinfeld realization. Furthermore in [13] Appendix A, we have obtained a realization of $U_{q,p}(\widehat{\mathfrak{g}})$ in terms of the Drinfeld generators in $U_q(\widehat{\mathfrak{g}})$ and a Heisenberg algebra $\mathbb{C}[P_{\alpha_j}, e^{Q_{\alpha_j}} (j \in I)]^1$. Note that such realization is well-defined in the p -adic topology. Hence applying the isomorphism between the Drinfeld-Jimbo realization of $U_q(\widehat{\mathfrak{g}})$ in terms of the Chevalley generators and the Drinfeld realization of the same algebra in terms of the Drinfeld generators, one can expect the isomorphism

$$U_{q,p}(\widehat{\mathfrak{g}}) \cong \mathcal{B}_{q,\lambda(r^*,P)}(\widehat{\mathfrak{g}}) \# \mathbb{C}[\mathcal{R}_Q] \quad (3.21)$$

as an associative algebra. Here $\lambda(r^*, P)$ is given in Appendix A. In fact one can derive the same *RLL*-relations as (3.19)-(3.20) by using the universal *R*-matrix of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$. There the $\mathcal{L}^\pm(u)$

¹The Heisenberg generators $P_{\alpha_j}, Q_{\alpha_j}$ are related to P_j, Q_j in [13] by $P_{\alpha_j} = d_j P_j, Q_{\alpha_j} = -2Q_j$, respectively.

operators are the elements in $\text{End}_{\mathbb{C}} V \otimes \mathcal{B}_{q, \lambda(r^*, P)}(\widehat{\mathfrak{g}}) \sharp \mathbb{C}[\mathcal{R}_Q]$ and satisfy $\mathcal{L}^-(u) = \mathcal{L}^+(u + r^* + c/2)$. Then by assuming the Gauss decomposition such as (3.18) in $\mathcal{L}^\pm(u)$ and denoting their Gauss coordinates by $\mathcal{E}_{j,i}^+(u), \mathcal{F}_{i,j}^+(u), \mathcal{K}_j^+(u)$ one can show that for $1 \leq j \leq N-1$, the

$$\mathcal{E}_j(u) := -\frac{1}{a_{j+1,j}^*[1]^*} \left(\mathcal{E}_{j+1,j}^+(u - \frac{j}{2} + \frac{c}{2}) - \mathcal{E}_{j+1,j}^-(u - \frac{j}{2}) \right), \quad (3.22)$$

$$\mathcal{F}_j(u) := \frac{1}{a_{j,j+1}[1]} \left(\mathcal{F}_{j+1,j}^+(u - \frac{j}{2}) - \mathcal{F}_{j+1,j}^-(u - \frac{j}{2} + \frac{c}{2}) \right) \quad (3.23)$$

with

$$\frac{a_{j+1,j}^* a_{j,j+1}[1]}{q - q^{-1}} = 1$$

satisfy the same relations as the elliptic currents $E_j(u)$ and $F_j(u)$ in Proposition 3.4 [2, 24]. Note that the formulas in Definition 3.5 gives a solution to (3.22)-(3.23). However we have not yet succeeded to confirm similar formulas for the $j = N$ case due to a difficulty of extracting the relations for these half currents from the *RLL*-relations.

4 Hopf Algebroid Structure

In this section, we introduce an *H*-Hopf algebroid structure into the elliptic algebra $U_{q,p}(B_N^{(1)})$ and formulate it as an elliptic quantum group.

4.1 Definition of the *H*-Hopf algebroid

Let us recall some basic facts on the *H*-Hopf algebroid following the works of Etingof and Varchenko [5] and of Koelink and Rosengren [17].

Definition 4.1 (*H*-bialgebroid). *An H-bialgebroid is an H-algebra A equipped with two H-algebra homomorphisms $\Delta : A \rightarrow A \widetilde{\otimes} A$ (the comultiplication) and $\varepsilon : A \rightarrow \mathcal{D}$ (the counit) such that*

$$\begin{aligned} (\Delta \widetilde{\otimes} \text{id}) \circ \Delta &= (\text{id} \widetilde{\otimes} \Delta) \circ \Delta, \\ (\varepsilon \widetilde{\otimes} \text{id}) \circ \Delta &= \text{id} = (\text{id} \widetilde{\otimes} \varepsilon) \circ \Delta, \end{aligned}$$

under the identification (2.44).

Definition 4.2 (*H*-Hopf algebroid). *An H-Hopf algebroid is an H-bialgebroid A equipped with a \mathbb{C} -linear map $S : A \rightarrow A$ (the antipode), such that*

$$\begin{aligned} S(\mu_r(\widehat{f})a) &= S(a)\mu_l(\widehat{f}), \quad S(a\mu_l(\widehat{f})) = \mu_r(\widehat{f})S(a), \quad \forall a \in A, \widehat{f} \in \mathcal{M}_{\mathfrak{h}^*}, \\ m \circ (\text{id} \widetilde{\otimes} S) \circ \Delta(a) &= \mu_l(\varepsilon(a)1), \quad \forall a \in A, \\ m \circ (S \widetilde{\otimes} \text{id}) \circ \Delta(a) &= \mu_r(T_\alpha(\varepsilon(a)1)), \quad \forall a \in A_{\alpha\beta}, \end{aligned}$$

where $m : A \widetilde{\otimes} A \rightarrow A$ denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in \mathcal{M}_{H^*}$.

The H -algebra \mathcal{D} is an H -Hopf algebroid with $\Delta_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D} \widetilde{\otimes} \mathcal{D}$, $\varepsilon_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$, $S_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$ defined by

$$\begin{aligned}\Delta_{\mathcal{D}}(\widehat{f}T_{-\alpha}) &= \widehat{f}T_{-\alpha} \widetilde{\otimes} T_{-\alpha}, \\ \varepsilon_{\mathcal{D}} &= \text{id}, \quad S_{\mathcal{D}}(\widehat{f}T_{-\alpha}) = T_{\alpha}\widehat{f} = (T_{\alpha}\widehat{f})T_{\alpha}.\end{aligned}$$

4.2 The H -Hopf algebroid $U_{q,p}(B_N^{(1)})$

Now let us consider the H -Hopf algebroid structure on $U = U_{q,p}(B_N^{(1)})$. Let us consider the generating function of the L -operator matrix elements $L_{i,j}^+(u)$. We define two H -algebra homomorphisms, the co-unit $\varepsilon : U \rightarrow \mathcal{D}$ and the co-multiplication $\Delta : U \rightarrow U \widetilde{\otimes} U$ by

$$\varepsilon(L_{i,j}^+(u)) = \delta_{i,j} T_{Q_{\varepsilon_i}} \quad (n \in \mathbb{Z}), \quad \varepsilon(e^Q) = e^Q, \quad (4.1)$$

$$\varepsilon(\mu_l(\widehat{f})) = \varepsilon(\mu_r(\widehat{f})) = \widehat{f}T_0, \quad (4.2)$$

$$\Delta(L_{i,j}^+(u)) = \sum_k L_{i,k}^+(u) \widetilde{\otimes} L_{k,j}^+(z), \quad (4.3)$$

$$\Delta(e^Q) = e^Q \widetilde{\otimes} e^Q, \quad (4.4)$$

$$\Delta(\mu_l(\widehat{f})) = \mu_l(\widehat{f}) \widetilde{\otimes} 1, \quad \Delta(\mu_r(\widehat{f})) = 1 \widetilde{\otimes} \mu_r(\widehat{f}). \quad (4.5)$$

In fact, one can check that Δ preserves the RLL -relations (3.19)-(3.20).

Lemma 4.3. *The maps ε and Δ satisfy*

$$(\Delta \widetilde{\otimes} \text{id}) \circ \Delta = (\text{id} \widetilde{\otimes} \Delta) \circ \Delta, \quad (4.6)$$

$$(\varepsilon \widetilde{\otimes} \text{id}) \circ \Delta = \text{id} = (\text{id} \widetilde{\otimes} \varepsilon) \circ \Delta. \quad (4.7)$$

Combining this with the H -algebra structure, the set $(U, \Delta, \mathcal{M}_{H^*}, \mu_l, \mu_r, \varepsilon)$ is an H -bialgebroid.

From (4.3), one can derive the following coproduct formulas for the basic half currents.

Proposition 4.4.

$$\begin{aligned}\Delta(K_j^+(u)) &= K_{j+1}^+(u) \widetilde{\otimes} K_{j+1}^+(u) \left(1 + 1 \widetilde{\otimes} l_{j+1,j+1}^{+'}(u) + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} 1 + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} l_{j+1,j+1}^{+'}(u) \right) \\ &\quad \times \left(1 + \Delta(l_{j+1,j+1}^{+'}(u)) \right)^{-1} \\ &= \left(1 + \Delta(l_{j+1,j+1}^{+''}(u)) \right)^{-1} \left(1 + 1 \widetilde{\otimes} l_{j+1,j+1}^{+''}(u) + l_{j+1,j+1}^{+''}(u) \widetilde{\otimes} 1 + l_{j+1,j+1}^{+''}(u) \widetilde{\otimes} l_{j+1,j+1}^{+''}(u) \right) \\ &\quad \times K_{j+1}^+(u) \widetilde{\otimes} K_{j+1}^+(u),\end{aligned}$$

$$\begin{aligned}
\Delta(E_{j+1,j}^+(u)) &= \left(1 + \Delta(l_{j+1,j+1}^{+'}(u))\right) \\
&\times \left(1 + 1 \widetilde{\otimes} l_{j+1,j+1}^{+'}(u) + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} 1 + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} l_{j+1,j+1}^{+'}(u)\right)^{-1} \\
&\times \left(1 \widetilde{\otimes} E_{j+1,j}^+(u) + E_{j+1,j}^+(u) \widetilde{\otimes} K_{j+1}^+(u)^{-1} K_j^+(u) + E_{j+1,j}^+(u) \widetilde{\otimes} l_{j,j}^{+'}(u) \right. \\
&\quad + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} E_{j+1,j}^+(u) + 1 \widetilde{\otimes} l_{j+1,j}^{+'}(u) + l_{j+1,j}^{+'}(u) \widetilde{\otimes} K_{j+1}^+(u)^{-1} K_j^+(u) \\
&\quad \left. + l_{j+1,j}^{+'}(u) \widetilde{\otimes} l_{j,j}^{+'}(u) + l_{j+1,j+1}^{+'}(u) \widetilde{\otimes} l_{j+1,j}^{+'}(u) + \sum_{\substack{1 \leq k \leq -1 \\ \neq j, j+1}} \widehat{L}_{j+1,k}^{+'}(u) \widetilde{\otimes} \widehat{L}_{k,j}^{+'}(u)\right) \\
&\quad - \Delta(l_{j+1,j}^{+'}(u)) \\
\Delta(F_{j,j+1}^+(u)) &= \left(K_j^+(u) K_{j+1}^+(u)^{-1} \widetilde{\otimes} F_{j,j+1}^+(u) + F_{j,j+1}^+(u) \widetilde{\otimes} 1 + F_{j,j+1}^+(u) \widetilde{\otimes} l_{j+1,j+1}^{+''}(u) \right. \\
&\quad + l_{j,j}^{+''}(u) \widetilde{\otimes} F_{j,j+1}^+(u) + l_{j,j+1}^{+''}(u) \widetilde{\otimes} 1 + K_j^+(u) K_{j+1}^+(u)^{-1} \widetilde{\otimes} l_{j,j+1}^{+''}(u) \\
&\quad \left. + l_{j,j+1}^{+''}(u) \widetilde{\otimes} l_{j+1,j+1}^{+''}(u) + l_{j,j}^{+''}(u) \widetilde{\otimes} l_{j,j+1}^{+''}(u) + \sum_{\substack{1 \leq k \leq -1 \\ \neq j, j+1}} \widehat{L}_{j,k}^{+''}(u) \widetilde{\otimes} \widehat{L}_{k,j+1}^{+''}(u)\right) \\
&\times \left(1 + 1 \widetilde{\otimes} l_{j+1,j+1}^{+''}(u) + l_{j+1,j+1}^{+''}(u) \widetilde{\otimes} 1 + l_{j+1,j+1}^{+''}(u) \widetilde{\otimes} l_{j+1,j+1}^{+''}(u)\right)^{-1} \\
&\times \left(1 + \Delta(l_{j+1,j+1}^{+''}(u))\right) - \Delta(l_{j,j+1}^{+''}(u)).
\end{aligned}$$

Here $l_{k,l}^{+'}(u) = K_{j+1}^+(u)^{-1} l_{k,l}^+(u)$, $l_{k,l}^{+''}(u) = l_{k,l}^+(u) K_{j+1}^+(u)^{-1}$ and

$$\begin{aligned}
l_{j,j}^+(u) &= \sum_{j \prec k} F_{j,k}^+(u) K_k^+(u) E_{k,j}^+(u), \\
l_{i,j}^+(u) &= \sum_{j \prec k} F_{i,k}^+(u) K_k^+(u) E_{k,j}^+(u), \quad (i \prec j) \\
l_{j,i}^+(u) &= \sum_{j \prec k} F_{j,k}^+(u) K_k^+(u) E_{k,i}^+(u).
\end{aligned}$$

Now let us define formally an algebra antihomomorphism (the antipode) $S : U \rightarrow U$ by

$$\begin{aligned}
S(L^+(z)) &= L^+(z)^{-1}, \\
S(e^Q) &= e^{-Q}, \quad S(\mu_r(\widehat{f})) = \mu_l(\widehat{f}), \quad S(\mu_l(\widehat{f})) = \mu_r(\widehat{f}).
\end{aligned}$$

Then we have the following Lemma.

Lemma 4.5. *The map S satisfies the antipode axioms*

$$\begin{aligned}
m \circ (\text{id} \otimes S) \circ \Delta(x) &= \mu_l(\varepsilon(x)1), \quad \forall x \in U, \\
m \circ (S \otimes \text{id}) \circ \Delta(x) &= \mu_r(T_\alpha(\varepsilon(x)1)), \quad \forall x \in (U)_{\alpha\beta}.
\end{aligned}$$

From Lemmas 4.3 and 4.5, we have

Theorem 4.6. *The H -algebra $U_{q,p}(B_N^{(1)})$ equipped with (Δ, ε, S) is an H -Hopf algebroid.*

Definition 4.7. *We call the H -Hopf algebroid $(U_{q,p}(B_N^{(1)}), H, \mathcal{M}_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, S)$ the elliptic quantum group $U_{q,p}(B_N^{(1)})$.*

5 Representations

5.1 Dynamical representations

Let us consider a vector space \widehat{V} over $\mathbb{F} = \mathcal{M}_{H^*}$, which is H -diagonalizable, i.e.

$$\widehat{V} = \bigoplus_{\lambda, \mu \in H^*} \widehat{V}_{\lambda, \mu}, \quad \widehat{V}_{\lambda, \mu} = \{v \in \widehat{V} \mid q^{P+h} \cdot v = q^{<\lambda, P+h>} v, \quad q^P \cdot v = q^{<\mu, P>} v \quad \forall P+h, P \in H\}.$$

Let us define the H -algebra $\mathcal{D}_{H, \widehat{V}}$ of the \mathbb{C} -linear operators on \widehat{V} by

$$\begin{aligned} \mathcal{D}_{H, \widehat{V}} &= \bigoplus_{\alpha, \beta \in H^*} (\mathcal{D}_{H, \widehat{V}})_{\alpha, \beta}, \\ (\mathcal{D}_{H, \widehat{V}})_{\alpha, \beta} &= \left\{ X \in \text{End}_{\mathbb{C}} \widehat{V} \left| \begin{array}{l} f(P+h)X = Xf(P+h+<\alpha, P+h>), \\ f(P)X = Xf(P+<\beta, P>) \\ f(P), f(P+h) \in \mathbb{F}, \quad X \cdot \widehat{V}_{\lambda, \mu} \subseteq \widehat{V}_{\lambda+\alpha, \mu+\beta} \end{array} \right. \right\}, \\ \mu_l^{\mathcal{D}_{H, \widehat{V}}}(\widehat{f})v &= f(<\lambda, P+h>, p)v, \quad \mu_r^{\mathcal{D}_{H, \widehat{V}}}(\widehat{f})v = f(<\mu, P>, p^*)v, \quad \widehat{f} \in \mathbb{F}, \quad v \in \widehat{V}_{\lambda, \mu}. \end{aligned}$$

Definition 5.1. [5, 17, 23] *We define a dynamical representation of $U_{q,p}(B_N^{(1)})$ on \widehat{V} to be an H -algebra homomorphism $\pi : U_{q,p}(B_N^{(1)}) \rightarrow \mathcal{D}_{H, \widehat{V}}$. By the action of $U_{q,p}(B_N^{(1)})$ we regard \widehat{V} as a $U_{q,p}(B_N^{(1)})$ -module.*

Definition 5.2. *For $k \in \mathbb{C}$, we say that a $U_{q,p}(B_N^{(1)})$ -module has level k if c act as the scalar k on it.*

Definition 5.3. *Let $\mathcal{H}, \mathcal{N}_+, \mathcal{N}_-$ be the subalgebras of $U_{q,p}(B_N^{(1)})$ generated by c, d, K_i^{\pm} ($i \in I$), by $\alpha_{i,n}^{\vee}$ ($i \in I, n \in \mathbb{Z}_{>0}$), $e_{i,n}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$) $f_{i,n}$ ($i \in I, n \in \mathbb{Z}_{>0}$) and by $\alpha_{i,-n}^{\vee}$ ($i \in I, n \in \mathbb{Z}_{>0}$), $e_{i,-n}$ ($i \in I, n \in \mathbb{Z}_{>0}$), $f_{i,-n}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$), respectively.*

Definition 5.4. *For $k \in \mathbb{C}$, $\lambda \in \mathfrak{h}^*$ and $\mu \in H^*$, a (dynamical) $U_{q,p}(B_N^{(1)})$ -module $\widehat{V}(\lambda, \mu)$ is called the level- k highest weight module with the highest weight (λ, μ) , if there exists a vector $v \in \widehat{V}(\lambda, \mu)$ such that*

$$\begin{aligned} \widehat{V}(\lambda, \mu) &= U_{q,p}(B_N^{(1)}) \cdot v, \quad \mathcal{N}_+ \cdot v = 0, \\ c \cdot v &= kv, \quad f(P) \cdot v = f(<\mu, P>)v, \quad f(P+h) \cdot v = f(<\lambda, P+h>)v. \end{aligned}$$

5.2 Finite dimensional dynamical representation

We here give an elliptic and dynamical analogue of the evaluation representation associated with the vector representation of $B_N^{(1)}$. Let us consider $\widehat{V} = \bigoplus_{1 \leq m \leq -1} \mathbb{F}v_m \otimes 1$ and $\widehat{V}_z = \widehat{V}[z, z^{-1}]$.

Here $e^{Q_\alpha} \in \mathbb{C}[\mathcal{R}_Q]$ acts on $f(P_\beta)v \otimes 1$ by $e^{Q_\alpha}(f(P_\beta)v \otimes 1) = f(P_\beta - (\alpha, \beta))v \otimes 1$.

Theorem 5.5. *Let $E_{j,k}$ ($1 \leq j, k \leq -1$) denote the matrix units such that $E_{j,k}v_l = \delta_{k,l}v_j$. The following gives the $(2N+1)$ -dimensional dynamical representation of $U_{q,p}(B_N^{(1)})$ on \widehat{V}_z .*

$$\begin{aligned} \pi_z(e_j(w)) &= \left(\frac{(pq^2; p)_\infty}{(p; p)_\infty} E_{j,j+1} \delta\left(q^j \frac{z}{w}\right) + \frac{(pq^2; p)_\infty}{(p; p)_\infty} E_{-j-1, -j} \delta\left(q^{-j} \xi \frac{z}{w}\right) \right) e^{-Q_{\alpha_j}}, \\ \pi_z(f_j(w)) &= \frac{(pq^{-2}; p)_\infty}{(p; p)_\infty} E_{j+1, j} \delta\left(q^j \frac{z}{w}\right) + \frac{(pq^{-2}; p)_\infty}{(p; p)_\infty} E_{-j, -j-1} \delta\left(q^{-j} \xi \frac{z}{w}\right), \\ \pi_z(\psi_j^-(w, p)) &= q^{\pi(h_j)} e^{-Q_{\alpha_j}} \frac{\Theta_p(q^{j-2h_j^+} \frac{z}{w})}{\Theta_p(q^j \frac{z}{w})} \frac{\Theta_p(q^{-j-2h_j^-} \xi \frac{z}{w})}{\Theta_p(q^{-j} \xi \frac{z}{w})}, \\ \pi_z(\psi_j^+(w, p)) &= q^{-\pi(h_j)} e^{-Q_{\alpha_j}} \frac{\Theta_p(q^{-j+2h_j^+} \frac{w}{z})}{\Theta_p(q^{-j} \frac{w}{z})} \frac{\Theta_p(q^{j+2h_j^-} \xi^{-1} \frac{w}{z})}{\Theta_p(q^j \xi^{-1} \frac{w}{z})}, \quad (1 \leq j \leq N-1) \\ \pi_z(e_N(w)) &= \frac{(pq^2; p)_\infty}{(p; p)_\infty} [2]_N E_{N,0} \delta\left(q^N \frac{z}{w}\right) + \frac{(pq; p)_\infty (pq^{-2}; p)_\infty}{(p; p)_\infty (pq^{-1}; p)_\infty} E_{0,-N} \delta\left(q^{N-1} \frac{z}{w}\right), \\ \pi_z(f_N(w)) &= \frac{(pq^2; p)_\infty (pq^{-1}; p)_\infty}{(pq; p)_\infty (p; p)_\infty} [2]_N E_{-N,0} \delta\left(q^{N-1} \frac{z}{w}\right) + \frac{(pq^{-2}; p)_\infty}{(p; p)_\infty} E_{0,N} \delta\left(q^N \frac{z}{w}\right), \\ \pi_z(\psi_N^-(w)) &= q_N^{\pi(h_N)} e^{-Q_{\alpha_N}} \frac{\Theta_p(q^{N-\frac{1}{2}-\frac{3}{2}h_N^+} \frac{z}{w})}{\Theta_p(q^{N-\frac{1}{2}+\frac{1}{2}h_N^+} \frac{z}{w})} \frac{\Theta_p(q^{N-\frac{1}{2}-\frac{3}{2}h_N^-} \xi \frac{z}{w})}{\Theta_p(q^{N-\frac{1}{2}+\frac{1}{2}h_N^-} \xi \frac{z}{w})}, \\ \pi_z(\psi_N^+(w)) &= q_N^{-\pi(h_N)} e^{-Q_{\alpha_N}} \frac{\Theta_p(q^{-(N-\frac{1}{2})+\frac{3}{2}h_N^+} \frac{w}{z})}{\Theta_p(q^{-(N-\frac{1}{2})-\frac{1}{2}h_N^+} \frac{w}{z})} \frac{\Theta_p(q^{-(N-\frac{1}{2})+\frac{3}{2}h_N^-} \xi^{-1} \frac{w}{z})}{\Theta_p(q^{-(N-\frac{1}{2})-\frac{1}{2}h_N^-} \xi^{-1} \frac{w}{z})}. \end{aligned}$$

Here $\pi(h_j) = h_j^+ + h_j^-$, $h_j^+ = E_{j,j} - E_{j+1,j+1}$, $h_j^- = E_{-j-1, -j-1} - E_{-j, -j}$, $\pi(h_N) = 2(E_{N,N} - E_{-N, -N})$, $h_N^+ = E_{N,N} - E_{0,0}$, $h_N^- = E_{0,0} - E_{-N, -N}$.

Theorem 5.6. *In terms of the half currents the dynamical representation (π_z, \widehat{V}_z) is given as*

follows.

$$\begin{aligned}
\pi_z(K_{+j}^+(v)) &= \rho^+(v-u) \left\{ \frac{[v-u]}{[v-u+1]} \sum_{1 \preceq k \preceq j-1} E_{k,k} + E_{j,j} + \frac{[v-u-1]}{[v-u]} \sum_{j+1 \preceq k \neq -j \preceq -1} E_{k,k} \right. \\
&\quad \left. + \frac{[v-u-1][v-u+j+\eta-1]}{[v-u][v-u+j+\eta]} E_{-j,-j} \right\} e^{-Q_{\varepsilon_j}}, \\
\pi_z(K_{-j}^+(v)) &= \rho^+(v-u) \left\{ \frac{[v-u]}{[v-u+1]} \sum_{1 \preceq k \neq j \preceq -(j+1)} E_{k,k} + \frac{[v-u][v-u-j-\eta]}{[v-u+1][v-u-j+\frac{2N-1}{2}+1]} E_{j,j} \right. \\
&\quad \left. + \frac{[v-u-1]}{[v-u]} \sum_{-(j-1) \preceq k \preceq -1} E_{k,k} + E_{-j,-j} \right\} e^{Q_{\varepsilon_j}}, \\
\pi_z(K_0^+(v)) &= \rho^+(v-u) \left\{ \frac{[v-u]}{[v-u+1]} \sum_{1 \preceq k \preceq N} E_{k,k} + \frac{[v-u-1]}{[v-u]} \sum_{1 \preceq k \preceq N} E_{-k,-k} \right. \\
&\quad \left. + \frac{[v-u+\frac{1}{2}][v-u-1]}{[v-u-\frac{1}{2}][v-u+1]} E_{0,0} \right\}.
\end{aligned}$$

For $1 \preceq j \prec l \preceq N+1 \equiv 0$,

$$\begin{aligned}
\pi_z(E_{l,j}^+(v)) &= e^{Q_{\varepsilon_l}} \left\{ -E_{j,l} \frac{[v-u-P_{j,l}][1]}{[v-u][P_{j,l}]} + E_{-l,-j} \frac{[v-u+l-1+\eta-P_{j,l}][1]}{[v-u+l-1+\eta][P_{j,l}]} \prod_{m=j+1}^{l-1} \frac{[P_{j,m}+1]}{[P_{j,m}]} \right\} e^{-Q_{\varepsilon_j}}, \\
\pi_z(F_{j,l}^+(v)) &= E_{1,j} \frac{[v-u+P_{j,l}][1]}{[v-u][P_{j,l}]} - E_{-j,-l} \frac{[v-u+l-1+\eta+P_{j,l}][1]}{[v-u+l-1+\eta][P_{j,l}]} \prod_{m=j+1}^{l-1} \frac{[P_{j,m}-1]}{[P_{j,m}]}, \\
\pi_z(E_{-j,-l}^+(v)) &= e^{-Q_{\varepsilon_j}} \left\{ -E_{-l,-j} \frac{[v-u-P_{-l,-j}][1]}{[v-u][P_{-l,-j}]} + E_{j,l} \frac{[v-u-j-\eta-P_{-l,-j}][1]}{[v-u-j-\eta][P_{-l,-j}]} \prod_{m=j+1}^{l-1} \frac{[P_{-l,-m}+1]}{[P_{-l,-m}]} \right\} e^{Q_{\varepsilon_l}}, \\
\pi_z(F_{-l,-j}^+(v)) &= E_{-j,-l} \frac{[v-u+P_{-l,-j}][1]}{[v-u][P_{-l,-j}]} - E_{l,j} \frac{[v-u-j-\eta+P_{-l,-j}+\delta_{l,N+1}][1]}{[v-u-j-\eta][P_{-l,-j}+\delta_{l,N+1}]} \prod_{m=j+1}^{l-1} \frac{[P_{-l,-m}-1+\delta_{l,N+1}]}{[P_{-l,-m}+\delta_{l,N+1}]}
\end{aligned}$$

i) $j \prec k \prec N$,

$$\begin{aligned}
& \pi_z(E_{-k,j}^+(v)) \\
&= e^{-Q_{\varepsilon_k}} \left\{ -E_{j,-k} \frac{[v-u-P_{j,-k}][1]}{[v-u][P_{j,-k}]} \right. \\
& \quad \left. + E_{k,-j} \frac{[v-u-k-\eta-P_{j,-k}][1]}{[v-u-k-\eta][P_{j,-k}]} \frac{[P_j+1]}{[P_j]} \prod_{m=k+1}^N \frac{[P_{j,-m}+1]}{[P_{j,-m}]} \prod_{m=j+1}^N \frac{[P_{j,m}+1]}{[P_{j,m}]} \right\} e^{-Q_{\varepsilon_j}}, \\
& \pi_z(F_{j,-k}^+(v)) \\
&= E_{-k,j} \frac{[v-u+P_{j,-k}][1]}{[v-u][P_{j,-k}]} \\
& \quad - E_{-j,k} \frac{[v-u-k-\eta+P_{j,-k}][1]}{[v-u-k-\eta][P_{j,-k}]} \frac{[P_j-1]}{[P_j]} \prod_{m=k+1}^N \frac{[P_{j,-m}-1]}{[P_{j,-m}]} \prod_{m=j+1}^N \frac{[P_{j,m}-1]}{[P_{j,m}]}.
\end{aligned}$$

ii) $j = k \prec N$,

$$\begin{aligned}
& \pi_z(E_{-j,j}^+(v)) \\
&= e^{-Q_{\varepsilon_j}} E_{j,-j} \left\{ -\frac{[v-u-2P_j-1][1][v-u-j-\eta+1]}{[v-u][2P_j+1][v-u-j-\eta]} \right. \\
& \quad \left. + G_{P_j} \frac{[v-u-2P_j-j-\lambda][1]}{[v-u-j-\lambda][2P_j+1]} \prod_{m=1}^{j-1} \left(\frac{[P_{j,-m}]}{[P_{j,-m}+1]} \frac{[P_{j,m}]}{[P_{j,m}+1]} \right) \right\} e^{-Q_{\varepsilon_j}}, \\
& \pi_z(F_{j,-j}^+(v)) \\
&= E_{-j,j} \left\{ \frac{[v-u+2P_j-1][1][v-u-j-\eta+1]}{[v-u][2P_j-1][v-u-j-\eta]} \right. \\
& \quad \left. - G_{-P_j} \frac{[v-u+2P_j-j-\eta][1]}{[v-u-j-\eta][2P_j-1]} \prod_{m=1}^{j-1} \left(\frac{[P_{j,-m}]}{[P_{j,-m}-1]} \frac{[P_{j,m}]}{[P_{j,m}-1]} \right) \right\}.
\end{aligned}$$

iii) $k \prec j \preceq N+1 \equiv 0$,

$$\begin{aligned}
& \pi_z(E_{-k,j}^+(v)) \\
&= e^{-Q_{\varepsilon_k}} \left\{ -E_{j,-k} \frac{[v-u-P_{j,-k}][1]}{[v-u][P_{j,-k}]} \right. \\
& \quad \left. + E_{k,-j} \frac{[v-u-k-\eta-P_{j,-k}][1]}{[v-u-k-\eta][P_{j,-k}]} \frac{[P_j+1]}{[P_j]} \prod_{\substack{m=k+1 \\ \neq j}}^N \frac{[P_{j,-m}+1]}{[P_{j,-m}]} \prod_{m=j+1}^N \frac{[P_{j,m}+1]}{[P_{j,m}]} \right\} e^{-Q_{\varepsilon_j}}, \\
& \pi_z(F_{j,-k}^+(v)) \\
&= E_{-k,j} \frac{[v-u+P_{j,-k}][1]}{[v-u][P_{j,-k}]} \\
& \quad - E_{-j,k} \frac{[v-u-k-\eta+P_{j,-k}][1]}{[v-u-k-\eta][P_{j,-k}]} \frac{[P_j-1]}{[P_j]} \prod_{\substack{m=k+1 \\ \neq j}}^N \frac{[P_{j,-m}-1]}{[P_{j,-m}]} \prod_{m=j+1}^N \frac{[P_{j,m}-1]}{[P_{j,m}]}.
\end{aligned}$$

In addition, it is also worth to remark the following formulas.

$$\begin{aligned}
\pi_z(H_j^+(v)) &= \left\{ \frac{[v-u-\frac{j}{2}-1]}{[v-u-\frac{j}{2}]} E_{j,j} + \frac{[v-u-\frac{j}{2}-1]}{[v-u-\frac{j}{2}]} E_{j+1,j+1} \right. \\
&\quad \left. + \frac{[v-u+\eta+\frac{j}{2}+1]}{[v-u+\eta+\frac{j}{2}]} E_{-j-1,-j-1} + \frac{[v-u+\eta+\frac{j}{2}-1]}{[v-u+\eta+\frac{j}{2}]} E_{-j,-j} \right\} e^{-Q_{\alpha_j}} \\
&\quad (1 \leq j \leq N-1), \\
\pi_z(H_N^+(v)) &= \left\{ \frac{[v-u-\frac{N}{2}+1]}{[v-u-\frac{N}{2}]} E_{N,N} + \frac{[v-u-\frac{N}{2}-1][v-u-\frac{N}{2}-\frac{1}{2}]}{[v-u-\frac{N}{2}][v-u-\frac{N}{2}+\frac{1}{2}]} E_{0,0} \right. \\
&\quad \left. + \frac{[v-u-\frac{N}{2}-\frac{1}{2}]}{[v-u-\frac{N}{2}+\frac{1}{2}]} E_{-N,-N} \right\} e^{-Q_{\alpha_N}}, \\
\pi_z(H_j^-(v)) &= \left\{ \frac{[u-v+\frac{j}{2}-1]}{[u-v+\frac{j}{2}]} E_{j,j} + \frac{[u-v+\frac{j}{2}+1]}{[u-v+\frac{j}{2}]} E_{j+1,j+1} \right. \\
&\quad \left. + \frac{[u-v-\eta-\frac{j}{2}-1]}{[u-v-\eta-\frac{j}{2}]} E_{-j-1,-j-1} + \frac{[u-v-\eta-\frac{j}{2}+1]}{[u-v-\eta-\frac{j}{2}]} E_{-j,-j} \right\} e^{-Q_{\alpha_j}} \\
&\quad (1 \leq j \leq N-1), \\
\pi_z(H_N^-(v)) &= \left\{ \frac{[u-v+\frac{N}{2}-1]}{[u-v+\frac{N}{2}]} E_{N,N} + \frac{[u-v+\frac{N}{2}+\frac{1}{2}][u-v+\frac{N}{2}-1]}{[u-v+\frac{N}{2}-\frac{1}{2}][u-v+\frac{N}{2}]} E_{0,0} \right. \\
&\quad \left. + \frac{[u-v+\frac{N}{2}+\frac{1}{2}]}{[u-v+\frac{N}{2}-\frac{1}{2}]} E_{-N,-N} \right\} e^{-Q_{\alpha_N}}.
\end{aligned}$$

Remark. The statements in this theorem and the next one remain unchanged when one uses $\rho_0^+(u)$ and $\rho_0^{+*}(u)$ instead of $\rho^+(u)$ and $\rho^{+*}(u)$, respectively. See Sec.6

Combining the formulas in Theorem 5.5, it is not so hard to show the following.

Corollary 5.7.

$$\pi_z(\widehat{L}_{i,j}^+(v))_{k,l} = R^+(v-u, P)_{ik}^{jl}.$$

Proof. For example, for $-N \preceq -j \preceq -1$, we obtain

$$\begin{aligned}
\pi_z(L_{-j,-j}^+(v)) &= \pi_z \left(K_{-j}^+(v) + \sum_{-j \preceq -k \preceq -1} F_{-j,-k}^+(v) K_{-k}^+(v) E_{-k,-j}^+(v) \right) \\
&= \rho^+(v-u) \left\{ \bar{b}(v-u) \sum_{1 \preceq k \neq j \preceq -(j+1)} E_{k,k} + \sum_{-(j-1) \preceq -k \preceq -1} b(v-u, P_{-j,-k}) E_{-k,-k} \right. \\
&\quad \left. + \bar{d}(v-u, P_j, P_{-j}) E_{j,j} + E_{-j,-j} \right\} e^{Q_{\varepsilon_j}}.
\end{aligned}$$

Here we used the identity

$$\begin{aligned} \bar{d}(u, P_j, P_{-j}) &= \frac{[u]}{[u+1]} \left(\frac{[u-j-\eta]}{[u-j+1-\eta]} \right. \\ &\quad \left. - \sum_{-(j-1) \preceq -k \preceq -1} \frac{[u-k-\eta+P_{-j,-k}][u-k-\eta-P_{-j,-k}][1]^2}{[u-k-\eta][u-k+1-\eta][P_{-j,-k}]^2} \prod_{m=k+1}^{j-1} \frac{[P_{-j,-m}-1][P_{-j,-m}+1]}{[P_{-j,-m}]^2} \right). \end{aligned}$$

In addition, we have

$$\begin{aligned} \pi_z(L_{0,0}^+(v)) &= \pi_z \left(K_0^+(v) + \sum_{-N \preceq -k \preceq -1} F_{0,-k}^+(v) K_{-k}^+(v) E_{-k,0}^+(v) \right) \\ &= \rho^+(v-u) \left\{ \bar{b}(v-u) \sum_{1 \preceq k \preceq N} E_{k,k} + e_0(v-u, P) E_{0,0} \right. \\ &\quad \left. + \sum_{-N \preceq -k \preceq -1} b(v-u, P_{0,-k}) E_{-k,-k} \right\} e^{Q_{\varepsilon_j}} \end{aligned}$$

This is due to the identity

$$e_0(u, P) = \frac{[u-1][u+\frac{1}{2}]}{[u+1][u-\frac{1}{2}]} - \frac{[u]}{[u+1]} \sum_{-N \preceq -k \preceq -1} \frac{[u-k-\eta+\frac{1}{2}+P_k][u-k-\eta+\frac{1}{2}-P_k][1]^2}{[u-k-\eta][u-k+1-\eta][P_k+\frac{1}{2}][P_k-\frac{1}{2}]}.$$

Furthermore, the coefficient of $E_{1,-1}$ in

$$\begin{aligned} \pi_z(\widehat{L}_{-1,1}^+(v)) &= \pi_z(K_{-1}^+(v) E_{-1,1}^+(v)) \\ &= \rho^+(v-u) E_{1,-1} \left\{ -\frac{[1][v-u-2P_1-1]}{[v-u+1][2P_1+1]} + G_{P_1} \frac{[v-u][v-u-2P_1-1-\lambda][1]}{[v-u+1][v-u-\lambda][2P_1+1]} \right\} e^{-Q_{\varepsilon_1}} \end{aligned}$$

coincides with $e_1(v-u, P)$, and for $k \prec j \preceq N$ the coefficient of $E_{k,-j}$ in

$$\begin{aligned} \pi_z(L_{-k,j}^+(v)) &= \pi_z \left(K_{-k}^+(v) E_{-k,j}^+(v) + \sum_{-(k-1) \preceq -l \preceq -1} F_{-k,-l}^+(v) K_{-l}^+(v) E_{-l,j}^+(v) \right) \\ &= \rho^+(v-u) \left\{ \bar{c}(v-u, P_{j,-k}) E_{j,-k} \right. \\ &\quad \left. + E_{k,-j} G_{P_j} \frac{[v-u][v-u-1-\eta-P_{j,-k}][1]}{[v-u+1][v-u-\eta][P_{j,-k}+1]} \prod_{m=1}^{j-1} \frac{[P_{j,m}]}{[P_{j,m}+1]} \prod_{m=1}^{k-1} \frac{[P_{-k,-m}-1]}{[P_{-k,-m}]} \right\} e^{-Q_{\varepsilon_j}} \end{aligned}$$

coincides with $\bar{d}(v-u, P_k, P_j)$. \square

5.3 The level-1 representation

Next we consider level-1 representation of $U_{q,p}(B_N^{(1)})$. We follow the work [6]. Let e^{α_i} ($i \in I$) be the generators of the group algebra $\mathbb{C}[\mathcal{Q}]$ with the following central extension.

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j) + (\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} e^{\alpha_j} e^{\alpha_i}$$

Let us consider the Neveu-Schwartz (NS) fermion $\{\Psi_n|n \in \mathbb{Z} + \frac{1}{2}\}$ and the Ramond (R) fermion $\{\Psi_n|n \in \mathbb{Z}\}$ satisfying the following anti-commutation relations.

$$\{\Psi_m, \Psi_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$$

with $\mathcal{N} = 1/(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$. We define

$$\mathcal{F}^{NS} = \mathbb{C}[\Psi_{-\frac{1}{2}}, \Psi_{-\frac{3}{2}} \cdots], \quad \tilde{\mathcal{F}}^R = \mathbb{C}[\Psi_{-1}, \Psi_{-2}, \dots]$$

and their submodules $\mathcal{F}_{even}^{NS,R}$ (reps. $\mathcal{F}_{odd}^{NS,R}$) generated by the even (reps. odd) number of Ψ_{-m} 's. Due to the zero-mode Ψ_0 we have two degenerate vacuum states 1 and $\Psi_0 1$. We hence consider the extended space

$$\hat{\mathcal{F}}^R = \tilde{\mathcal{F}}^R \otimes \mathbb{C}^2$$

and realize the R -fermions by

$$\hat{\Psi}_m = \Psi_m \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (m \in \mathbb{Z}_{\neq 0}), \quad \hat{\Psi}_0 = \mathcal{N}^{\frac{1}{2}}(1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

Note that $\{\hat{\Psi}_m, \hat{\Psi}_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$. We set

$$\mathcal{F}^R = \mathcal{F}_{even}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathcal{F}_{odd}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of Ψ_m on \mathcal{F}^{NS} is given by

$$\Psi_{-m} \cdot u = \Psi_{-m} u, \quad \Psi_m \cdot u = \{\Psi_m, u\} \quad (m \in \frac{1}{2} + \mathbb{Z}_{\geq 0}),$$

where $u \in \mathcal{F}^{NS}$, whereas $\hat{\Psi}_m$ acts on \mathcal{F}^R as

$$\begin{aligned} \hat{\Psi}_{-m} \cdot u \otimes v &= \Psi_{-m} u \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), \quad \hat{\Psi}_0 \cdot u \otimes v = u \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v, \\ \hat{\Psi}_m \cdot u \otimes v &= \{\Psi_m, u\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), \end{aligned}$$

where $u \in \tilde{\mathcal{F}}^R$, $v \in \mathbb{C}^2$. We define the fermion fields $\Psi^{NS}(z)$ and $\Psi^R(z)$ by

$$\Psi^{NS}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \Psi_n z^{-n}, \quad \Psi^R(z) = \sum_{n \in \mathbb{Z}} \hat{\Psi}_n z^{-n}.$$

Then we have the operator product expansions

$$\begin{aligned}\Psi(z)\Psi(w) &=:\Psi(z)\Psi(w): + \langle \Psi(z)\Psi(w) \rangle, \\ \langle \Psi(z)\Psi(w) \rangle &= \begin{cases} \frac{(zw)^{1/2}(z-w)}{(z-qw)(z-q^{-1}w)} & \text{for NS} \\ \mathcal{N} \frac{(z-w)(z+w)}{(z-qw)(z-q^{-1}w)} & \text{for R.} \end{cases}\end{aligned}$$

Now we define

$$\begin{aligned}W(\Lambda_0) &= \mathcal{F}_{even}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1}, \\ W(\Lambda_1) &= \mathcal{F}_{even}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1} \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0], \\ W(\Lambda_N) &= \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}] e^{\bar{\Lambda}_N} \cong \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_N} \oplus \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1 + \bar{\Lambda}_N},\end{aligned}$$

where \mathcal{Q}_0 denotes the sublattice of \mathcal{Q} generated by the long roots. For generic $\mu \in \mathfrak{h}^*$ and $a = 0, 1, N$, we set

$$\widehat{V}(\Lambda_a + \mu, \mu) = \mathcal{F}_{\alpha,1} \otimes_{\mathbb{C}} (\mathbb{F} \otimes_{\mathbb{C}} W(\Lambda_a)) \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q].$$

Then we have the following decomposition.

$$\widehat{V}(\Lambda_a + \mu, \mu) = \bigoplus_{\gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}} \bigoplus_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0 + \mathbb{C}\delta}} \mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu),$$

where

$$\begin{aligned}\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{even}^{NS} \otimes e^{\gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{odd}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{even}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{odd}^{NS} \otimes e^{\gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_N, \gamma, \kappa}(\Lambda_N, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_N + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\ \mathcal{F}_{\Lambda_N - \alpha_N, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_N + \bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}.\end{aligned}$$

Theorem 5.8. [6] *The three spaces $\widehat{V}(\Lambda_a + \mu, \mu)$ ($a = 0, 1, N$) give the level-1 irreducible $U_{q,p}(\hat{B}_l^{(1)})$ -modules with the highest weight $(\Lambda_a + \mu, \mu)$, where the highest weight vectors are given by $1 \otimes 1 \otimes 1 \otimes e^{Q_{\bar{\mu}}}$ for $\widehat{V}(\Lambda_0 + \mu, \mu)$, $1 \otimes 1 \otimes e^{\bar{\Lambda}_1} \otimes e^{Q_{\bar{\mu}}}$ for $\widehat{V}(\Lambda_1 + \mu, \mu)$ and $1 \otimes 1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes e^{\bar{\Lambda}_N} \otimes e^{Q_{\bar{\mu}}}$ for $\widehat{V}(\Lambda_N + \mu, \mu)$, respectively. The action of the elliptic currents on $\widehat{V}(\Lambda_a + \mu, \mu)$ ($a = 0, 1, N$)*

is given as follows.

$$E_j(v) = : \exp \left\{ - \sum_{n \neq 0} \frac{1}{[n]_q} \alpha_{j,n} w^{-n} \right\} : e^{\alpha_j} w^{h_{\alpha_j}} e^{-Q_{\alpha_j}} w^{-\frac{P_{\alpha_j}-1}{r^*}}, \quad (5.1)$$

$$E_N(v) = \frac{1}{\mathcal{N}^{\frac{1}{2}}} : \exp \left\{ - \sum_{n \neq 0} \frac{1}{[n]_q} \alpha_{N,n} w^{-n} \right\} : \Psi(w) e^{\alpha_N} w^{h_{\alpha_N} + \frac{1}{2}} e^{-Q_{\alpha_N}} w^{-\frac{P_{\alpha_N}-1/2}{r^*}}, \quad (5.2)$$

$$F_j(v) = : \exp \left\{ \sum_{n \neq 0} \frac{1}{[n]_q} \frac{1-p^{*n}}{1-p^n} \alpha_{j,n} (q^{-1}w)^{-n} \right\} : e^{-\alpha_j} w^{-h_{\alpha_j}} w^{\frac{P_{\alpha_j}+h_{\alpha_j}-1}{r}}, \quad (5.3)$$

$$F_N(v) = \frac{1}{\mathcal{N}^{\frac{1}{2}}} : \exp \left\{ \sum_{n \neq 0} \frac{1}{[n]_q} \frac{1-p^{*n}}{1-p^n} \alpha_{N,n} (q^{-1}w)^{-n} \right\} : \Psi(w) e^{-\alpha_N} w^{-h_{\alpha_N} + \frac{1}{2}} w^{\frac{P_{\alpha_N}+h_{\alpha_N}-1/2}{r}}, \quad (5.4)$$

$w = q^{2v}$, ($1 \leq j \leq N-1$) together with $H_j^{\pm}(v)$, $K_j^{\pm}(v)$ in Sec. 3.2.

6 Vertex Operators of $U_{q,p}(B_N^{(1)})$

In this section we discuss the type I and II vertex operators of the $U_{q,p}(B_N^{(1)})$ -modules. Through this section, we use $\rho_0^+(u)$ as the prefactor of the R -matrix i.e.

$$R^+(u, s) = \rho_0^+(u) \bar{R}^+(u, s).$$

In addition, we often use the following component form of the RLL -relation (3.19).

$$\begin{aligned} \sum_{i', j'} R^+(u, P+h)_{i,j}^{i' j'} \hat{L}_{i', i''}^+(u_1) \hat{L}_{j', j''}^+(u_2) \\ = \sum_{i', j'} \hat{L}_{j, j'}^+(u_2) \hat{L}_{i, i'}^+(u_1) R^{*+}(u, P - (\pi(h))_{i', i''} - (\pi(h))_{j', j''})_{i' j'}^{i'' j''}. \end{aligned} \quad (6.1)$$

Here the components of the R^{*+} -matrix is evaluated in the same way as (3.3). For example, the $(j_1, j_2), (j_1, j_2)$ ($j_1 \prec j_2$) component is given by

$$b^*(u, P_{j_1, j_2} - (\pi(h_{j_1, j_2}))_{j_1, j_1} - (\pi(h_{j_1, j_2}))_{j_2, j_2}),$$

where $P_{j_1, j_2} = P_{\epsilon_{j_1}} - P_{\epsilon_{j_2}}$, $h_{j_1, j_2} = h_{\epsilon_{j_1}} - h_{\epsilon_{j_2}}$ and $\pi(h_{\epsilon_j}) = E_{j, j} - E_{-j, -j}$ ($1 \leq j \leq N$).

6.1 Definition

The type I and II vertex operators are the intertwiners of the $U_{q,p}(B_N^{(1)})$ -modules of the form

$$\hat{\Phi}(u) : \hat{V}(\lambda, \mu) \rightarrow \hat{V}(\lambda', \mu) \otimes \hat{V}_z, \quad (6.2)$$

$$\hat{\Psi}^*(u) : \hat{V}_z \otimes \hat{V}(\lambda, \mu) \rightarrow \hat{V}(\lambda, \mu'), \quad (6.3)$$

where $\lambda, \lambda' \in \mathfrak{h}^*, \mu, \mu' \in H^*, z = q^{2u}$. The \widehat{V}_z denote the $(2N+1)$ -dimensional dynamical evaluation module of $U_{q,p}(B_N^{(1)})$ given in Theorem 5.5 and 5.6, and $\widehat{V}(\lambda, \mu)$ denote the level- k highest weight $U_{q,p}(B_N^{(1)})$ -module with highest weight (λ, μ) . The level-1 case is given in Theorem 5.8. The vertex operators satisfy the intertwining relations with respect to the comultiplication Δ given in (4.3)

$$\Delta(x)\widehat{\Phi}(u) = \widehat{\Phi}(u)x, \quad (6.4)$$

$$x\widehat{\Psi}^*(u) = \widehat{\Psi}^*(u)\Delta(x), \quad \forall x \in U_{q,p}(B_N^{(1)}). \quad (6.5)$$

These intertwining relations are equivalent to the following relations [22]:

$$\widehat{\Phi}^{(23)}(u_2)\widehat{L}^{+(12)}(u_1) = R^{+(13)}(u_1 - u_2, P + h)\widehat{L}^{+(12)}(u_1)\widehat{\Phi}^{(23)}(u_2), \quad (6.6)$$

$$\widehat{L}^{+(13)}(u_1)\widehat{\Psi}^{*(23)}(u_2) = \widehat{\Psi}^{*(23)}(u_2)\widehat{L}^{+(13)}(u_1)R^{+*(12)}(u_1 - u_2, P - h^{(1)} - h^{(2)}). \quad (6.7)$$

The relation (6.6) (resp. (6.7)) should be understood on $\widehat{V}_{z_1} \otimes \widehat{V}(\lambda, \mu)$ (resp. $\widehat{V}_{z_1} \otimes \widehat{V}_{z_2} \otimes \widehat{V}(\lambda, \mu)$). These relations are also expected [13] from the quasi-Hopf algebra formulation of the face type elliptic quantum group $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ [12] by using the connection given in Appendix A.

We define the components of the vertex operators by

$$\widehat{\Phi}(u + \frac{1}{2}) = \sum_{1 \preceq m \preceq -1} \Phi_m(u) \otimes v_m, \quad \widehat{\Psi}^*(u)(v_m \otimes u) = \Psi_m^*\left(u - \frac{c}{2}\right) u, \quad (6.8)$$

where $v_m \in \widehat{V}_z$, $u \in \widehat{V}(\lambda, \mu)$, and the matrix elements of the L -operator $\widehat{L}^+(u)$ by

$$\widehat{L}^+(u)v_m = \sum_{1 \preceq k \preceq -1} L_{k,m}^+(u)v_k. \quad (6.9)$$

Using these and Corollary 5.7, the intertwining relations (6.6), (6.7) are rewritten as follows:

$$\Phi_m(u_2)L_{k,j}^+(u_1) = \sum_{1 \preceq m', k' \preceq -1} R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{km}^{k'm'} L_{k',j}^+(u_1)\Phi_{m'}(u_2), \quad (6.10)$$

$$L_{k,j}^+(u_1)\Psi_m^*(u_2) = \sum_{1 \preceq j', m' \preceq -1} \Psi_{m'}^*(u_2)L_{k,j'}^+(u_1)R^{+*}(u_1 - u_2 - \frac{c}{2}, P - h^{(1)} - h^{(2)})_{j'm'}^{jm}. \quad (6.11)$$

Proposition 6.1. *Let the half currents $E_{l,k}^+(u)$ and $F_{k,l}^+(u)$ ($1 \preceq k \prec l \preceq -1$) take their form as given in Definition 3.5 and Appendix B. Assume that the top components $\Phi_{-1}(u)$ and $\Psi_{-1}^*(u)$ satisfy the following conditions:*

i) $K_{-1}^+(u_1)\Phi_{-1}(u_2)$ does not have a pole at $u_1 - u_2 = -\frac{3}{2}$

ii) $\Psi_{-1}(u_2)K_{-1}^+(u_1)$ does not have a pole at $u_1 - u_2 = \frac{c-2}{2} + r^*$.

Then the sufficient conditions for (6.10) and (6.11) are given as follows. For the type I,

$$\Phi_k(u_2) = F_{k,-1}^+(u_2 - \frac{1}{2})\Phi_{-1}(u_2) \quad (1 \preceq k \preceq -2), \quad (6.12)$$

and

$$\Phi_{-1}(u_2)K_{-1}^+(u_1) = \rho_0^+(u_1 - u_2 + \frac{1}{2})K_{-1}^+(u_1)\Phi_{-1}(u_2), \quad (6.13)$$

$$[\Phi_{-1}(u_2), P_l] = 0, \quad [\Phi_{-1}(u_2), E_l(u_1)] = 0. \quad (1 \leq l \leq N), \quad (6.14)$$

$$\Phi_{-1}(u_2)(P+h)_{k,-1} = ((P+h)_{k,-1} - 1)\Phi_{-1}(u_2), \quad (6.15)$$

$$\Phi_{-1}(u_2)F_1(u_1) = \frac{[u_2 - u_1 - \eta]}{[u_2 - u_1 - \eta - 1]}F_1(u_2)\Phi_{-1}(u_1), \quad (6.16)$$

$$\Phi_{-1}(u_2)F_j(u_1) = F_j(u_1)\Phi_{-1}(u_2) \quad (2 \leq j \leq N). \quad (6.17)$$

For the type II,

$$\Psi_k^*(u_2) = \Psi_{-1}^*(u_2)E_{-1,k}^+(u_2 + \frac{c}{2} + r^*) \quad (1 \preceq k \preceq -2), \quad (6.18)$$

and

$$K_{-1}^+(u_1)\Psi_{-1}^*(u_2) = \Psi_{-1}^*(u_2)K_{-1}^+(u_1)\rho_0^{+*}(u_1 - u_2 - \frac{c}{2}), \quad (6.19)$$

$$[\Psi_{-1}^*(u_2), (P+h)_l] = 0, \quad [\Psi_{-1}^*(u_2), F_l(u_1)] = 0 \quad (1 \leq l \leq N), \quad (6.20)$$

$$P_{j,-1}\Psi_{-1}^*(u_2) = \Psi_{-1}^*(u_2)(P_{j,-1} + 1) \quad (j \prec -1), \quad (6.21)$$

$$E_1(u_1)\Psi_{-1}^*(u_2) = \frac{[u_2 - u_1 - \eta + \frac{1}{2}]^*}{[u_2 - u_1 - \eta - \frac{1}{2}]^*}\Psi_{-1}^*(u_2)E_1(u_1), \quad (6.22)$$

$$E_j(u_1)\Psi_{-1}^*(u_2) = \Psi_{-1}^*(u_2)E_j(u_1) \quad (2 \leq j \leq N). \quad (6.23)$$

Proof. We consider the type I case only. The type II case can be proved similarly. From the component $k = m (\neq 0)$ in (6.10), we have

$$\Phi_m(u_2)L_{m,j}^+(u_1) = \rho_0^+(u_1 - u_2 + \frac{1}{2})L_{m,j}^+(u_1)\Phi_m(u_2). \quad (6.24)$$

In particular, the component $m = j = -1$ of (6.24) is

$$\Phi_{-1}(u_2)K_{-1}^+(u_1) = \rho_0^+(u_1 - u_2 + \frac{1}{2})K_{-1}^+(u_1)\Phi_{-1}(u_2). \quad (6.25)$$

Note the formula

$$\rho_0^+(u) = \frac{[u+1]}{\varphi(u)}, \quad (6.26)$$

$$\varphi(u) = q^{-1}z^{\frac{1}{r}}[u-1] \frac{\{\xi^2 z\}\{z\}\{\xi q^2 z\}\{\xi q^{-2} z\}}{\{\xi z\}^2\{\xi^2 q^2 z\}\{q^{-2} z\}} \frac{\{p\xi/z\}^2\{p\xi^2 q^2/z\}\{pq^{-2}/z\}}{\{p\xi^2/z\}\{p/z\}\{p\xi q^2/z\}\{p\xi q^{-2}/z\}}. \quad (6.27)$$

Then from the assumption i), (6.25) implies that $\Phi_{-1}(u_2)K_{-1}^+(u_1)$ has a zero at $u_1 - u_2 = -\frac{3}{2}$. We will check these points for the level-1 representation.

In addition, from the component $m = -1 \succ j$ of (6.24), and putting the definition $L_{-1,j}^+(u) = K_{-1}^+(u)E_{-1,j}^+(u)$, we have

$$\Phi_{-1}(u_2)E_{-1,j}^+(u_1) = E_{-1,j}^+(u_1)\Phi_{-1}(u_2). \quad (6.28)$$

From the conjectural expressions for $E_{-1,j}^+(u)$ in Appendix B, the sufficient conditions for (6.28) are

$$[\Phi_{-1}(u_2), P_l] = 0, \quad [\Phi_{-1}(u_2), E_l(u_1)] = 0. \quad (1 \leq l \leq N) \quad (6.29)$$

Next, the component $k \neq \pm m (\neq 0)$, $k \prec m$ in (6.10) is

$$\begin{aligned} \Phi_m(u_2)L_{k,j}^+(u_1) &= \rho_0^+(u_1 - u_2 + \frac{1}{2}) \left\{ b(u_1 - u_2 + \frac{1}{2}, (P+h)_{k,m})L_{k,j}^+(u_1)\Phi_m(u_2) \right. \\ &\quad \left. + c(u_1 - u_2 + \frac{1}{2}, (P+h)_{k,m})L_{m,j}^+(u_1)\Phi_k(u_2) \right\}. \end{aligned} \quad (6.30)$$

Then putting the definition $L_{-1,-1}^+(u) = K_{-1}^+(u)$ and $L_{k,-1}^+(u) = F_{k,-1}^+(u)K_{-1}^+(u)$ in the case $k \prec m = j = -1$ of (6.30), we have

$$\begin{aligned} &\Phi_{-1}(u_2)F_{k,-1}^+(u_1)K_{-1}^+(u_1) \\ &= \rho_0^+(u_1 - u_2 + \frac{1}{2}) \left\{ \frac{[(P+h)_{k,-1} + 1][(P+h)_{k,-1} - 1][u_1 - u_2 + \frac{1}{2}]}{[(P+h)_{k,-1}]^2[u_1 - u_2 + \frac{3}{2}]} F_{k,-1}^+(u_1)K_{-1}^+(u_1)\Phi_{-1}(u_2) \right. \\ &\quad \left. + \frac{[1][(P+h)_{k,-1} + u_1 - u_2 + \frac{1}{2}]}{[(P+h)_{k,-1}][u_1 - u_2 + \frac{3}{2}]} K_{-1}^+(u_1)\Phi_k(u_2) \right\}. \end{aligned} \quad (6.31)$$

Putting $u_1 - u_2 = -\frac{3}{2}$, the left hand side of (6.31) vanishes. Then we obtain for $2 \preceq k \preceq -2$

$$\begin{aligned} \Phi_k(u_2) &= K_{-1}^+(u_1)^{-1} \frac{[(P+h)_{k,-1} + 1]}{[(P+h)_{k,-1}]} F_{k,-1}^+(u_1)K_{-1}^+(u_1)\Phi_{-1}(u_2) \\ &= F_{k,-1}^+(u_1 + 1)\Phi_{-1}(u_2) \\ &= F_{k,-1}^+(u_2 - \frac{1}{2})\Phi_{-1}(u_2). \end{aligned} \quad (6.32)$$

The remaining component $\Phi_1(u)$ is also obtained from $\Phi_{-1}(u)$ as follows. From the component $m = j = -1$ in (6.10), we have

$$\begin{aligned} \Phi_{-1}(u_2)F_{1,-1}^+(u_1)K_{-1}^+(u_1) &= R^+(u_1 - u_2 + \frac{1}{2}, P+h)_{1,-1}^{1,-1} F_{1,-1}^+(u_1)K_{-1}^+(u_1)\Phi_{-1}(u_2) \\ &\quad + R^+(u_1 - u_2 + \frac{1}{2}, P+h)_{1,-1}^{-1,1} K_{-1}^+(u_1)\Phi_1(u_2) \\ &\quad + \sum_{2 \preceq l \preceq -2} R^+(u_1 - u_2 + \frac{1}{2}, P+h)_{1,-1}^{-l,l} F_{l,-1}^+(u_1)K_{-1}^+(u_1)F_{l,-1}(u_2)\Phi_{-1}(u_2). \end{aligned}$$

Using (6.32) and the component $(i, j) = (1, -1)$, $(i'', j'') = (-1, -1)$ of (6.1), we obtain

$$\begin{aligned}
& \Phi_{-1}(u_2)F_{1,-1}^+(u_1)K_{-1}^+(u_1) \\
&= R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{1,-1}^{-11} K_{-1}^+(u_1)\Phi_1(u_2) \\
&+ \rho_0^+(u_1 - u_2 + \frac{1}{2})K_{-1}^+(u_2)F_{1,-1}^+(u_1)K_{-1}^+(u_2)^{-1}K_{-1}^+(u_1)\Phi_{-1}(u_2) \\
&- R^+(u_1 - u_2 + \frac{1}{2}, P + h)_{1,-1}^{-11} K_{-1}^+(u_1)F_{1,-1}^+(u_2)\Phi_{-1}(u_2).
\end{aligned}$$

Then again setting $u_1 - u_2 = -\frac{3}{2}$, the left hand side and the second term in right hand side vanish. Then we obtain

$$\Phi_1(u_2) = F_{1,-1}^+(u_2 - \frac{1}{2})\Phi_{-1}(u_2). \quad (6.33)$$

Combining (6.32) and (6.33), we obtain (6.12).

Furthermore, substituting (6.12) into (6.31), we obtain the sufficient conditions (6.15)-(6.17).

Lemma 6.2. *For $1 \preceq k \preceq -2$, we have*

$$\begin{aligned}
\Phi_{-1}(u_2)F_{k,-1}(u_1 - \frac{1}{2}) &= K_{-1}^+(u_2 - \frac{1}{2})F_{k,-1}^+(u_1 - \frac{1}{2})K_{-1}^+(u_2 - \frac{1}{2})^{-1}\Phi_{-1}(u_2), \\
E_{-1,k}^+(u_1 + \frac{c}{2})\Psi_{-1}^*(u_2) &= \Psi_{-1}^*(u_2)K_{-1}^+(u_2 + \frac{c}{2})^{-1}E_{-1,k}^+(u_1 + \frac{c}{2})K_{-1}^+(u_2 + \frac{c}{2}).
\end{aligned}$$

Proof. From the component $m = -1$, $j = -1$ in the intertwining relation (6.10), we have

$$\begin{aligned}
& \Phi_{-1}(u_2)F_{k,-1}^+(u_1)K_{-1}^+(u_1) \\
&= \rho_0^+(u_1 - u_2 + \frac{1}{2})b(u_1 - u_2 + \frac{1}{2}, (P + h)_{k,-1})F_{k,-1}^+(u_1)K_{-1}^+(u_1)\Phi_{-1}(u_2) \\
&+ \rho_0^+(u_1 - u_2 + \frac{1}{2})c(u_1 - u_2 + \frac{1}{2}, (P + h)_{k,-1})K_{-1}^+(u_1)\Phi_k(u_2). \quad (6.34)
\end{aligned}$$

Using (6.12), we have

$$\begin{aligned}
& \Phi_{-1}(u_2)F_{k,-1}^+(u_1)K_{-1}^+(u_1) \\
&= \rho_0^+(u_1 - u_2 + \frac{1}{2})b(u_1 - u_2 + \frac{1}{2}, (P + h)_{k,-1})F_{k,-1}^+(u_1)K_{-1}^+(u_1)\Phi_{-1}(u_2) \\
&+ \rho_0^+(u_1 - u_2 + \frac{1}{2})c(u_1 - u_2 + \frac{1}{2}, (P + h)_{k,-1})K_{-1}^+(u_1)F_{k,-1}^+(u_2 - \frac{1}{2})\Phi_{-1}(u_2). \quad (6.35)
\end{aligned}$$

From (6.25), we have

$$\begin{aligned}
& \Phi_{-1}(u_2)F_{k,-1}^+(u_1 - \frac{1}{2}) \\
&= b(u_1 - u_2, (P + h)_{k,-1})F_{k,-1}^+(u_1 - \frac{1}{2})\Phi_{-1}(u_2) \\
&+ c(u_1 - u_2, (P + h)_{k,-1})K_{-1}^+(u_1 - \frac{1}{2})F_{k,-1}^+(u_2 - \frac{1}{2})K_{-1}^+(u_1 - \frac{1}{2})^{-1}\Phi_{-1}(u_2). \quad (6.36)
\end{aligned}$$

Then it is sufficient to show

$$\begin{aligned} & b(u_1 - u_2, (P + h)_{k,-1}) F_{k,-1}^+(u_1) + c(u_1 - u_2, (P + h)_{k,-1}) K_{-1}^+(u_1) F_{k,-1}^+(u_2) K_{-1}^+(u_1)^{-1} \\ &= K_{-1}^+(u_2) F_{k,-1}^+(u_1) K_{-1}^+(u_2)^{-1}. \end{aligned} \quad (6.37)$$

This is nothing but the component $(i, j) = (-1, -1)$, $(i'', j'') = (k, -1)$ of (6.1). \square

6.2 Level one vertex operators and commutation relations

Next we consider a free field realization of the vertex operators fixing the representation level $c = 1$.

From the sufficient conditions obtained in Proposition 6.1, we can determine the free field realizations of vertex operators as follows:

Proposition 6.3. *The highest components of the type I and type II vertex operators $\Phi_{-1}(u)$ and $\Psi_{-1}^*(u)$ are realized in terms of the free field by*

$$\Phi_{-1}(u) =: \exp \left\{ \sum_{m \neq 0} (q^m - q^{-m}) \frac{1 - p^{*m}}{1 - p^m} \mathcal{E}_m^{-1} (q^{-3} \xi z)^{-m} \right\} : e^{\epsilon_1} (q^{-1} \xi z)^{h_{\epsilon_1}} (q^{-1} \xi z)^{-\frac{1}{r}(P+h)_{\epsilon_1}}, \quad (6.38)$$

$$\Psi_{-1}^*(u) =: \exp \left\{ - \sum_{m \neq 0} (q^m - q^{-m}) \mathcal{E}_m^{-1} (q^{-1} \xi z)^{-m} \right\} : e^{-\epsilon_1} e^{Q_{\epsilon_1}} (\xi z)^{-h_{\epsilon_1}} (\xi z)^{\frac{1}{r^*} P_{\epsilon_1}}. \quad (6.39)$$

These realizations satisfy the assumptions i) and ii) in Proposition 6.1.

Proof. By straightforward calculations, we can show that (6.38) (resp. (6.39)) satisfies the sufficient conditions (6.25), (6.29), and (6.15)-(6.17) (resp. (6.19)-(6.23)). \square

Theorem 6.4. *The free field realizations of the type I $\Phi_j(u)$ and the type II $\Psi_j^*(u)$ vertex operators satisfy the following commutation relations:*

$$\Phi_{j_2}(u_2) \Phi_{j_1}(u_1) = \sum_{j'_1, j'_2=1}^{-1} R(u_1 - u_2, P + h)_{j'_1 j'_2}^{j_1 j_2} \Phi_{j'_1}(u_1) \Phi_{j'_2}(u_2), \quad (6.40)$$

$$\Psi_{j_1}^*(u_1) \Psi_{j_2}^*(u_2) = \sum_{j'_1, j'_2=1}^{-1} \Psi_{j'_2}^*(u_2) \Psi_{j'_1}^*(u_1) R^*(u_1 - u_2, P - h^{(1)} - h^{(2)})_{j'_1 j'_2}^{j_1 j_2}, \quad (6.41)$$

$$\Phi_j(u_1) \Psi_k^*(u_2) = \chi(u_1 - u_2) \Psi_k^*(u_2) \Phi_j(u_1). \quad (6.42)$$

Here we set

$$R(u, P) = \mu(u) \bar{R}^+(u, P), \quad R^*(u, P) = \mu^*(u) \bar{R}^{+*}(u, P) \quad (6.43)$$

with

$$\mu(u) = z^{-1+\frac{1}{r}} \frac{\{p\xi^2 q^{-2} z\} \{p\xi z\} \{\xi z\} \{q^2 z\} \{p\xi q^{-2}/z\} \{p/z\} \{\xi^2/z\} \{\xi q^2/z\}}{\{p\xi q^{-2} z\} \{pz\} \{\xi^2 z\} \{\xi q^2 z\} \{p\xi^2 q^{-2}/z\} \{p\xi/z\} \{\xi/z\} \{q^2/z\}}, \quad (6.44)$$

$$\mu^*(u) = \mu(u)|_{r \rightarrow r^*} \quad (6.45)$$

and

$$\chi(u) = \frac{\Theta_{\xi^2}(z) \Theta_{\xi^2}(q^{-2} \xi z)}{\Theta_{\xi^2}(\xi z) \Theta_{\xi^2}(q^{-2} \xi^2 z)}. \quad (6.46)$$

Proof. Let us show the commutation relation of the type I vertex operators (6.40). For $j_1 = j_2 = -1$, the equation

$$\Phi_{-1}(u_2) \Phi_{-1}(u_1) = \mu(u_1 - u_2) \Phi_{-1}(u_1) \Phi_{-1}(u_2) \quad (6.47)$$

can be shown by straightforward calculation with the use of the free field realization (6.38).

For $1 \preceq j_1, j_2 \preceq -2$, using (6.12), (6.47) and Lemma 6.2, the equation (6.40) is reduced to the following equation:

$$\begin{aligned} & F_{j_2, -1}^+(u_2) K_{-1}^+(u_2) F_{j_1, -1}^+(u_1) K_{-1}^+(u_2)^{-1} \\ &= \sum_{j'_1, j'_2=1}^{-1} \bar{R}^+(u, P+h)_{j_1 j_2}^{j'_1 j'_2} F_{j'_1, -1}^+(u_1) K_{-1}^+(u_1) F_{j'_2, -1}^+(u_2) K_{-1}^+(u_1)^{-1}, \end{aligned} \quad (6.48)$$

where $u = u_1 - u_2$. From the component $(i, j) = (-1, -1)$, $(i'', j'') = (j_1, j_2)$ of (6.1), we have

$$\begin{aligned} & \sum_{j'_1, j'_2=1}^{-1} R^+(u, P+h)_{j_1 j_2}^{j'_1 j'_2} F_{j'_1, -1}^+(u_1) K_{-1}^+(u_1) F_{j'_2, -1}^+(u_2) K_{-1}^+(u_2) \\ &= F_{j_2, -1}^+(u_2) K_{-1}^+(u_2) F_{j_1, -1}^+(u_1) K_{-1}^+(u_1) \tilde{\rho}^{*+}(u). \end{aligned} \quad (6.49)$$

Multiplying the above equation by

$$\tilde{\rho}^+(u)^{-1} K_{-1}^+(u_2)^{-1} K_{-1}^+(u_1)^{-1} = K_{-1}^+(u_1)^{-1} K_{-1}^+(u_2)^{-1} \tilde{\rho}^{*+}(u)^{-1} \quad (6.50)$$

from the right, we obtain the desired equation (6.48).

For $j_1 = -1, 1 \preceq j_2 \preceq -2$, using (6.12), (6.47) and Lemma 6.2, the equation (6.40) is reduced to the following equation of the half currents:

$$\begin{aligned} & F_{j_2, -1}^+(u_2) \quad (6.51) \\ &= \bar{R}^+(u, P+h)_{-1 j_2}^{-1-1} + \sum_{j'_2=1}^{-2} \bar{R}^+(u, P+h)_{-1 j_2}^{-1 j'_2} K_{-1}^+(u_1 - \frac{1}{2}) F_{j'_2, -1}^+(u_2 - \frac{1}{2}) K_{-1}^+(u_1 - \frac{1}{2})^{-1} \\ &+ \sum_{j'_1=1}^{-2} \bar{R}^+(u, P+h)_{-1 j_2}^{j'_1 -1} F_{j'_1, -1}^+(u_1 - \frac{1}{2}) \\ &+ \sum_{j'_1, j'_2=1}^{-2} \bar{R}^+(u, P+h)_{-1 j_2}^{j'_1 j'_2} F_{j'_1, -1}^+(u_1 - \frac{1}{2}) K_{-1}^+(u_1 - \frac{1}{2}) F_{j'_2, -1}^+(u_2 - \frac{1}{2}) K_{-1}^+(u_1 - \frac{1}{2})^{-1}. \end{aligned}$$

From the component $(i, j) = (-1, -1)$, $(i'', j'') = (-1, j_2)$ of (6.1), we have

$$\begin{aligned}
& R^+(u, P+h)_{-1j_2}^{-1-1} K_{-1}^+(u_1) K_{-1}^+(u_2) \\
& + \sum_{j'=1}^{-2} R^+(u, P+h)_{-1j_2}^{-1j'} K_{-1}^+(u_1) F_{j',-1}^+(u_2) K_{-1}^+(u_2) \\
& + \sum_{i'=1}^{-2} R^+(u, P+h)_{-1j_2}^{i'-1} F_{i',-1}^+(u_1) K_{-1}^+(u_1) K_{-1}^+(u_2) \\
& + \sum_{i',j'=1}^{-2} R^+(u, P+h)_{-1j_2}^{i'j'} F_{i',-1}^+(u_1) K_{-1}^+(u_1) F_{j',-1}^+(u_2) K_{-1}^+(u_2) \\
& = F_{j_2,-1}^+(u_2) K_{-1}^+(u_2) K_{-1}^+(u_1) \rho^{+*}(u).
\end{aligned} \tag{6.52}$$

Multiplying the above equation by (6.50) from the right, we obtain the desired equation (6.51).

The case $1 \leq j_1 \leq -2$, $j_2 = -1$ can be proved in the same manner.

Similarly, one can prove the commutation relation of the type II vertex operators (6.41).

Next, let us consider the relation (6.42). The case $j = k = -1$ is a direct consequence from Proposition 6.3. The cases $j = -1$ or $k = -1$ can be shown as follows: consider the case $k = -1$ for instance. By (6.12),

$$\begin{aligned}
\Phi_j(u_1) \Psi_{-1}^*(u_2) &= F_{j,-1}^+(u_1 - \frac{1}{2}) \Phi_{-1}(u_1) \Psi_{-1}^*(u_2) \\
&= F_{j,-1}^+(u_1 - \frac{1}{2}) \chi(u_1 - u_2) \Psi_{-1}^*(u_2) \Phi_{-1}(u_1) \\
&= \chi(u_1 - u_2) \Psi_{-1}^*(u_2) F_{j,-1}^+(u_1 - \frac{1}{2}) \Phi_{-1}(u_1) \\
&= \chi(u_1 - u_2) \Psi_{-1}^*(u_2) \Phi_j(u_1).
\end{aligned} \tag{6.53}$$

Then the general case is proved as follows. Since both $\widehat{\Phi}(u_1) \widehat{\Psi}^*(u_2)$ and $(\widehat{\Psi}^*(u_2) \otimes \text{id})(\text{id} \otimes \widehat{\Phi}(u_1))$ commute with $\Delta(x)$ ($\forall x \in U_{q,p}(B_N^{(1)})$), they act as scalars on the irreducible module $V_{z_2} \otimes \widehat{V}(\lambda)$. In order to compare the scalars, we will see their actions on $v_{-1} \otimes |\lambda\rangle \in V_{z_2} \otimes \widehat{V}(\lambda)$.

$$\begin{aligned}
\widehat{\Phi}(u_1) \widehat{\Psi}^*(u_2)(v_{-1} \otimes |\lambda\rangle) &= \widehat{\Phi}(u_1) \Psi_{-1}^*(u_2 - \frac{c}{2}) |\lambda\rangle \\
&= \sum_j \Phi_j(u_1) \Psi_{-1}^*(u_2 - \frac{c}{2}) |\lambda\rangle \otimes v_j \\
&= \chi(u_1 - u_2 + \frac{1}{2}) \sum_j \Psi_{-1}^*(u_2 - \frac{c}{2}) \Phi_j(u_1 + \frac{1}{2}) |\lambda\rangle \otimes v_j.
\end{aligned} \tag{6.54}$$

Here the last equality follows from (6.53). On the other hand

$$\begin{aligned}
(\widehat{\Psi}^*(u_2) \otimes \text{id})(\text{id} \otimes \widehat{\Phi}(u_1))(v_{-1} \otimes |\lambda\rangle) &= (\widehat{\Psi}^*(u_2) \otimes \text{id})(v_{-1} \otimes \sum_j \Phi_j(u_1) |\lambda\rangle \otimes v_j) \\
&= \Psi_{-1}^*(u_2 - \frac{c}{2}) \sum_j \Phi_j(u_1 + \frac{1}{2}) |\lambda\rangle \otimes v_j.
\end{aligned} \tag{6.55}$$

Comparing (6.54) and (6.55), we get

$$\widehat{\Phi}(u_1)\widehat{\Psi}^*(u_2) = \chi(u_1 - u_2 + \frac{1+c}{2})(\widehat{\Psi}^*(u_2) \otimes \text{id})(\text{id} \otimes \widehat{\Phi}(u_1)). \quad (6.56)$$

Hence comparing the components of the both sides, and changing variables $u_1 + \frac{1}{2} \rightarrow u_1$, $u_2 - \frac{c}{2} \rightarrow u_2$, we obtain

$$\Phi_j(u_1)\Psi_k^*(u_2) = \chi(u_1 - u_2)\Psi_k^*(u_2)\Phi_j(u_1). \quad (6.57)$$

□

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A Relation to the Quasi-Hopf Formulation $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$

A.1 Definition of $\mathcal{B}_{q,\lambda}(B_N^{(1)})$

Let $U_q = U_q(B_N^{(1)})$ be the Drinfeld-Jimbo affine quantum group [3, 9]. Namely, $U_q(B_N^{(1)})$ is a quasi-triangular Hopf algebra realized by the Chevalley generators and equipped with the standard coproduct Δ_0 , counit ε , antipode S and universal R matrix \mathcal{R} . Our conventions on the coalgebra structure follows [12]. Let \mathfrak{h} and $\bar{\mathfrak{h}}$ be the Cartan subalgebras as in Sec.2.1. We denote a basis and its dual basis of \mathfrak{h} by $\{\hat{h}_l\}$ and $\{\hat{h}^l\}$, respectively. More explicitly, they are given by $\{\hat{h}_l\} = \{d, c, h_j\}$ and $\{\hat{h}^l\} = \{c, d, h^j\}$ ($1 \leq j \leq N$), where $\{h_j\}$ and $\{h^j\}$ are a basis and a dual basis of $\bar{\mathfrak{h}}$.

The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ is a quasi-Hopf deformation of $U_q(B_N^{(1)})$ by the face type twistor $F(\lambda)$ ($\lambda \in \mathfrak{h}$). The twistor $F(\lambda)$ is an invertible element in $U_q \otimes U_q$ satisfying

$$(\text{id} \otimes \varepsilon)F(\lambda) = 1 = F(\lambda)(\varepsilon \otimes \text{id}), \quad (\text{A.1})$$

$$F^{(12)}(\lambda)(\Delta_0 \otimes \text{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\text{id} \otimes \Delta_0)F(\lambda). \quad (\text{A.2})$$

where $\lambda = \sum_l \lambda_l \hat{h}^l$ ($\lambda_l \in \mathbb{C}$), $\lambda + h^{(1)} = \sum_l (\lambda_l + \hat{h}_l^{(1)}) \hat{h}^l$ and $\hat{h}_l^{(1)} = \hat{h}_l \otimes 1 \otimes 1$. An explicit construction of the twistor $F(\lambda)$ is given in [12]. Then we define a new coproduct by

$$\Delta_\lambda(x) = F(\lambda) \Delta_0(x) F(\lambda)^{-1} \quad \forall x \in U_q(B_N^{(1)}). \quad (\text{A.3})$$

Δ_λ satisfies a weaker coassociativity

$$(\text{id} \otimes \Delta_\lambda) \Delta_\lambda(x) = \Phi(\lambda) (\Delta_\lambda \otimes \text{id}) \Delta_\lambda(x) \Phi(\lambda)^{-1} \quad \forall x \in U_q(B_N^{(1)}), \quad (\text{A.4})$$

$$\Phi(\lambda) = F^{(23)}(\lambda) F^{(23)}(\lambda + h^{(1)})^{-1}. \quad (\text{A.5})$$

The universal R -matrix is also deformed to

$$\mathcal{R}(\lambda) = F^{(21)}(\lambda) \mathcal{R} F^{(12)}(\lambda)^{-1}. \quad (\text{A.6})$$

Definition A.1. [12] *The face type elliptic quantum group $\mathcal{B}_{q,\lambda}(B_N^{(1)})$ is a quasi-triangular quasi-Hopf algebra $(U_q(B_N^{(1)}), \Delta_\lambda, \varepsilon, S, \Phi(\lambda), \alpha, \beta, \mathcal{R}(\lambda))$, where α, β are defined by*

$$\alpha = \sum_i S(k_i) l_i, \quad \beta = \sum_i m_i S(n_i). \quad (\text{A.7})$$

Here we set $\sum_i k_i \otimes l_i = F(\lambda)^{-1}$, $\sum_i m_i \otimes n_i = F(\lambda)$.

The new universal R matrix $\mathcal{R}(\lambda)$ satisfies the dynamical Yang-Baxter equation.

$$\mathcal{R}^{(12)}(\lambda + h^{(3)}) \mathcal{R}^{(13)}(\lambda) \mathcal{R}^{(23)}(\lambda + h^{(1)}) = \mathcal{R}^{(23)}(\lambda) \mathcal{R}^{(13)}(\lambda + h^{(2)}) \mathcal{R}^{(12)}(\lambda). \quad (\text{A.8})$$

In [21], we derived vector representations of $\mathcal{R}(\lambda)$ for $\hat{\mathfrak{g}} = A_N^{(1)}, B_N^{(1)}, C_N^{(1)}, D_N^{(1)}$ and found that if we parametrize $\lambda \in \mathfrak{h}^*$ as $\lambda = \lambda(r^*, P) = (r^* + h^\vee)d + \sum_{j=1}^N (P_{\alpha'_j} + 1) \bar{\Lambda}_j$ with α'_j being the simple roots of the dual Lie algebra $\bar{\mathfrak{g}}^\vee$ of $\bar{\mathfrak{g}}$, the vector representation of $\mathcal{R}(\lambda)$ coincides with the corresponding face weight derived by Jimbo, Okado and Miwa [10]. In particular, for the $B_N^{(1)}$ type, if we set

$$\begin{aligned} \mathcal{R}^{+*}(z, P) &= (\pi_{V,z} \otimes \pi_{V,1}) \left(\text{Ad } z^{-\frac{\bar{\theta}(\lambda)}{r}} \otimes \text{id} \right) \left(z^{\frac{\bar{T}}{r}} q^{c \otimes d + d \otimes c} \mathcal{R}(\lambda) \right), \\ \mathcal{R}^+(z, P + h) &= (\pi_{V,z} \otimes \pi_{V,1}) \left(\text{Ad } z^{-\frac{\bar{\theta}(\lambda)}{r}} \otimes \text{id} \right) \left(z^{\frac{\bar{T}}{r}} q^{c \otimes d + d \otimes c} \mathcal{R}(\lambda + h) \right), \end{aligned}$$

with

$$\begin{aligned} \bar{\theta}(\lambda) &= -\bar{\lambda} + \bar{\rho} - \frac{1}{2} \sum_j \bar{h}_j \bar{h}^j, \\ \bar{T} &= \sum_j \bar{h}_j \otimes \bar{h}^j \end{aligned}$$

for $\lambda = \lambda(r^*, P)$, then $\mathcal{R}^{+*}(z, P)$ and $\mathcal{R}^+(z, P + h)$ coincide with (3.3) up to a gauge transformation. Moreover we define the L operators of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{g}})$ by

$$\begin{aligned}\mathcal{L}^+(z, P) &= (\pi_{V,z} \otimes \text{id}) \left(\text{Ad } z^{-\frac{\bar{\theta}(\lambda)}{r}} \otimes \text{id} \right) \left(z^{\frac{\bar{T}}{r}} q^{c \otimes d + d \otimes c} \mathcal{R}(\lambda) \right), \\ \mathcal{L}^-(z, P) &= (\pi_{V,z} \otimes \text{id}) \left(\text{Ad } z^{-\frac{\bar{\theta}(\lambda)}{r}} \otimes \text{id} \right) \left(z^{\frac{\bar{T}}{r}} \mathcal{R}^{(21)}(\lambda)^{-1} q^{-c \otimes d - d \otimes c} \right).\end{aligned}$$

Note that $\mathcal{L}^+(z, P)$ and $\mathcal{L}^-(z, P)$ are not independent: we have

$$\mathcal{L}^-(z, P) = \mathcal{L}^+(zp^*q^c, P). \quad (\text{A.9})$$

In addition, if we define

$$\begin{aligned}\mathcal{R}^{-*}(z, P) &= (\pi_{V,z_1} \otimes \pi_{V,z_2}) \mathcal{R}^{(21)}(\lambda)^{-1} q^{-c \otimes d - d \otimes c}, \\ \mathcal{R}^-(z, P + h) &= (\pi_{V,z_1} \otimes \pi_{V,z_2}) \mathcal{R}^{(21)}(\lambda + h)^{-1} q^{-c \otimes d - d \otimes c},\end{aligned}$$

Then we have $\mathcal{R}^{-*}(z, P) = \mathcal{R}^{+*}(zp^*q^c, P)$, $\mathcal{R}^-(z, P + h) = \mathcal{R}^+(zpq^{-c}, P + h)$. Combining these formulas we obtain from (A.8) the following dynamical RLL relations [13]

$$\mathcal{R}^\pm(z, P + h) \mathcal{L}^\pm(z_1, P) \mathcal{L}^\mp(z_2, P + h^{(1)}) = \mathcal{L}^\pm(z_2, P) \mathcal{L}^\pm(z_1, P + h^{(2)}) \mathcal{R}^{\pm*}(z, P), \quad (\text{A.10})$$

$$\mathcal{R}^\pm(zq^{\pm c}, P + h) \mathcal{L}^\pm(z_1, P) \mathcal{L}^\mp(z_2, P + h^{(1)}) = \mathcal{L}^\mp(z_2, P) \mathcal{L}^\pm(z_1, P + h^{(2)}) \mathcal{R}^{\pm*}(zq^{\mp c}, P). \quad (\text{A.11})$$

Furthermore define

$$\widehat{\mathcal{L}}^\pm(z) = \mathcal{L}^\pm(z, P) e^{-\sum_j \pi_V(h_{\bar{\varepsilon}_j}) \otimes Q_{\bar{\varepsilon}_j}} \in \mathcal{B}_{q,\lambda(r^*, P)}(\widehat{\mathfrak{g}})[[z, z^{-1}]] [z^{\pm \frac{1}{r^*}}, z^{\pm \frac{1}{r}}] \sharp \mathbb{C}[\mathcal{R}_Q],$$

where $\pi_V(h_{\varepsilon_j}) = E_{jj} - E_{-j-j}$ for the case $B_N^{(1)}$. Then one can verify that $\widehat{\mathcal{L}}^\pm(z)$ satisfy the RLL relations

$$\mathcal{R}^\pm(z, P + h) \widehat{\mathcal{L}}^\pm(z_1) \widehat{\mathcal{L}}^\mp(z_2) = \widehat{\mathcal{L}}^\pm(z_2) \widehat{\mathcal{L}}^\pm(z_1) \mathcal{R}^{\pm*}(z, P), \quad (\text{A.12})$$

$$\mathcal{R}^\pm(zq^{\pm c}, P + h) \widehat{\mathcal{L}}^\pm(z_1) \widehat{\mathcal{L}}^\mp(z_2) = \widehat{\mathcal{L}}^\mp(z_2) \widehat{\mathcal{L}}^\pm(z_1) \mathcal{R}^{\pm*}(zq^{\mp c}, P). \quad (\text{A.13})$$

These RLL -relations coincide with (3.19) and (3.20).

B Integral Expressions for the Half Currents

For $X = E, F$, let us denote by $[X_{j_1}(v_1) \cdots X_{j_m}(v_m)]$ the product of the elliptic currents $X_{j_1}(v_1), \dots, X_{j_m}(v_m)$ where all the zero-modes, $w_j^{-\frac{P_{\alpha_j}-1}{r^*}}$ of $E_j(v_j)$ ($j = 1, \dots, N-1$), $w_N^{-\frac{P_{\alpha_N}-1/2}{r^*}}$

of $E_N(v_N)$, $w_j^{-\frac{(P+h)\alpha_j-1}{r}}$ of $F_j(v_j)$ ($j = 1, \dots, N-1$) and $w_N^{-\frac{(P+h)\alpha_N-1/2}{r}}$ of $F_N(v_N)$, are normally ordered, i.e. they are moved to the right of all $e_j(w_j)$ and $f_j(w_j)$ by using (2.5) and (2.6).

i) The $j \prec k \prec N$ case:

$$\begin{aligned}
& [E_j(v_j)E_{j+1}(v_{j+1}) \cdots E_k(v_k) \cdots E_N(v_N)E_N(v'_N) \cdots E_k(v'_k)] \\
&= e_j(v_j)e_{j+1}(v_{j+1}) \cdots e_k(v_k) \cdots e_N(v_N)e_N(v'_N) \cdots e_k(v'_k) \prod_{m=j}^{k-2} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_{k-1}^{-\frac{P\alpha_{k-1}+1}{r^*}} \cdot w_k^{-\frac{P\alpha_k-1}{r^*}} \\
&\quad \times \prod_{m=k+1}^{N-1} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_N^{-\frac{P\alpha_N-1/2}{r^*}} w_N'^{-\frac{P\alpha_N+1/2}{r^*}} \prod_{m=k+1}^{N-1} w_m'^{-\frac{P\alpha_m}{r^*}} \cdot w_k'^{-\frac{P\alpha_k-1}{r^*}}, \tag{B.1}
\end{aligned}$$

$$\begin{aligned}
& [E_k(v'_k)E_{k+1}(v'_{k+1}) \cdots E_N(v'_N)E_N(v_N) \cdots E_k(v_k) \cdots E_j(v_j)] \\
&= e_k(v'_k)e_{k+1}(v'_{k+1}) \cdots e_N(v'_N)e_N(v_N) \cdots e_k(v_k) \cdots e_j(v_j) \prod_{m=k}^{N-1} w_m'^{-\frac{P\alpha_m}{r^*}} \cdot w_N'^{-\frac{P\alpha_N-1/2}{r^*}} w_N^{-\frac{P\alpha_N+1/2}{r^*}} \\
&\quad \times \prod_{m=j+1}^{N-1} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_j^{-\frac{P\alpha_j-1}{r^*}}. \tag{B.2}
\end{aligned}$$

ii) The $j = k \prec N$ case:

$$\begin{aligned}
& [E_j(v_j)E_{j+1}(v_{j+1}) \cdots E_N(v_N)E_N(v'_N) \cdots E_j(v'_j)] \\
&= e_j(v_j)e_{j+1}(v_{j+1}) \cdots e_N(v_N)e_N(v'_N) \cdots e_j(v'_j) w_j^{-\frac{P\alpha_j-1}{r^*}} \prod_{m=j+1}^{N-1} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_N^{-\frac{P\alpha_N-1/2}{r^*}} w_N'^{-\frac{P\alpha_N+1/2}{r^*}} \\
&\quad \times \prod_{m=j+1}^{N-1} w_m'^{-\frac{P\alpha_m}{r^*}} \cdot w_j'^{-\frac{P\alpha_j-1}{r^*}}, \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
& [E_j(v'_j)E_{j+1}(v'_{j+1}) \cdots E_N(v_N)E_N(v_N) \cdots E_j(v_j)] \\
&= e_j(v'_j)e_{j+1}(v'_{j+1}) \cdots e_N(v_N)e_N(v_N) \cdots e_j(v_j) w_j'^{-\frac{P\alpha_j-1}{r^*}} \prod_{m=j+1}^{N-1} w_m'^{-\frac{P\alpha_m}{r^*}} \cdot w_N'^{-\frac{P\alpha_N-1/2}{r^*}} w_N^{-\frac{P\alpha_N+1/2}{r^*}} \\
&\quad \times \prod_{m=j+1}^{N-1} w_m^{-\frac{P\alpha_m}{r^*}} \cdot w_j^{-\frac{P\alpha_j-1}{r^*}}. \tag{B.4}
\end{aligned}$$

iii) The $k \prec j \prec N$ case:

$$\begin{aligned}
& [E_j(v_j)E_{j+1}(v_{j+1}) \cdots E_N(v_N)E_N(v'_N) \cdots E_j(v'_j) \cdots E_k(v'_k)] \\
& = e_j(v_j)e_{j+1}(v_{j+1}) \cdots e_N(v_N)e_N(v'_N) \cdots e_j(v'_j) \cdots e_k(v'_k) \prod_{m=j}^{N-1} w_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w_N^{-\frac{P_{\alpha N}-1/2}{r^*}} w'_N^{-\frac{P_{\alpha N}+1/2}{r^*}} \\
& \quad \times \prod_{m=k+1}^{N-1} w'_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w'_k^{-\frac{P_{\alpha k}-1}{r^*}}, \tag{B.5}
\end{aligned}$$

$$\begin{aligned}
& [E_j(v'_k)E_{k+1}(v'_{k+1}) \cdots E_j(v'_j) \cdots E_N(v'_N)E_N(v_N) \cdots E_k(v_j)] \\
& = e_j(v'_k)e_{k+1}(v'_{k+1}) \cdots e_j(v'_j) \cdots e_N(v'_N)e_N(v_N) \cdots e_k(v_j) \prod_{m=k}^{j-2} w'_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w'_{j-1}^{-\frac{P_{\alpha j-1}+1}{r^*}} \cdot w'_j^{-\frac{P_{\alpha j}-1}{r^*}} \\
& \quad \times \cdot \prod_{m=j+1}^{N-1} w'_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w'_N^{-\frac{P_{\alpha N}-1/2}{r^*}} w_N^{-\frac{P_{\alpha N}+1/2}{r^*}} \prod_{m=j+1}^{N-1} w_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w_j^{-\frac{P_{\alpha j}-1}{r^*}}. \tag{B.6}
\end{aligned}$$

iv) The $k \prec j = N$ case:

$$\begin{aligned}
& [E_N(v_N)E_N(v'_N) \cdots E_k(v'_k)] \\
& = e_N(v_N)e_N(v'_N) \cdots e_j(v'_j) \cdots e_k(v'_k) w_N^{-\frac{P_{\alpha N}-1/2}{r^*}} w'_N^{-\frac{P_{\alpha N}+1/2}{r^*}} \prod_{m=k+1}^{N-1} w'_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w'_k^{-\frac{P_{\alpha k}-1}{r^*}}, \tag{B.7}
\end{aligned}$$

$$\begin{aligned}
& [E_k(v'_k) \cdots E_N(v'_N)E_N(v_N)] \\
& = e_k(v'_k) \cdots e_N(v'_N)e_N(v_N) \prod_{m=k}^{N-2} w'_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w'_{N-1}^{-\frac{P_{\alpha N-1}+1}{r^*}} \cdot w'_N^{-\frac{P_{\alpha k}-3/2}{r^*}} w_N^{-\frac{P_{\alpha N}-1/2}{r^*}}. \tag{B.8}
\end{aligned}$$

v) The $j \prec k = N$ case:

$$\begin{aligned}
& [E_j(v_j) \cdots E_N(v_N)E_N(v'_N)] \\
& = e_k(v_j) \cdots e_N(v_N)e_N(v'_N) \prod_{m=j}^{N-2} w_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w_{N-1}^{-\frac{P_{\alpha N-1}+1}{r^*}} \cdot w_N^{-\frac{P_{\alpha k}-3/2}{r^*}} w'_N^{-\frac{P_{\alpha N}-1/2}{r^*}}, \tag{B.9}
\end{aligned}$$

$$\begin{aligned}
& [E_N(v'_N)E_N(v_N) \cdots E_k(v_j)] \\
& = e_N(v'_N)e_N(v_N) \cdots e_k(v_j) w'_N^{-\frac{P_{\alpha N}-1/2}{r^*}} w_N^{-\frac{P_{\alpha N}+1/2}{r^*}} \prod_{m=j+1}^{N-1} w_m^{-\frac{P_{\alpha m}}{r^*}} \cdot w_j^{-\frac{P_{\alpha j}-1}{r^*}}. \tag{B.10}
\end{aligned}$$

vi) The $j = k = N$ case:

$$[E_N(v_N)E_N(v'_N)] = e_N(v_N)e_N(v'_N) w_N^{-\frac{P_{\alpha k}-3/2}{r^*}} w'_N^{-\frac{P_{\alpha N}-1/2}{r^*}}, \tag{B.11}$$

$$[E_N(v'_N)E_N(v_N)] = e_N(v'_N)e_N(v_N) w'_N^{-\frac{P_{\alpha k}-3/2}{r^*}} w_N^{-\frac{P_{\alpha N}-1/2}{r^*}}. \tag{B.12}$$

The $F_j(v_j)$'s counterpart of the product $[E_{j_1}(v_1) \cdots E_{j_m}(v_m)]$ is obtained by replacing $e_{j_k}(v_k)$ with $f_{j_k}(v_k)$, w_j with w_j^{-1} and r^* with r .

Conjecture B.1. For $1 \preceq j \prec l \preceq 0$,

$$\begin{aligned}
F_{j,l}^+(v) &= a_{j,l} \oint_{C_{j,l}^+} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j)] f_{j,l}^+(v, v_j \cdots, v_{l-1}, P+h) \\
&\quad + a_{j,l} \oint_{C_{j,l}^-} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [F_j(v_j) F_{j+1}(v_{j+1}) \cdots F_{l-1}(v_{l-1})] f_{j,l}^-(v, v_j \cdots, v_{l-1}, P+h), \\
F_{-l,-j}^+(v) &= a_{-l,-j} \oint_{C_{-l,-j}^+} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [F_j(v_j) F_{j+1}(v_{j+1}) \cdots F_{l-1}(v_{l-1})] f_{-l,-j}^+(v, v_j \cdots, v_{l-1}, P+h) \\
&\quad + a_{-l,-j} \oint_{C_{-l,-j}^-} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [F_{l-1}(v_{l-1}) F_{l-2}(v_{l-2}) \cdots F_j(v_j)] f_{-l,-j}^-(v, v_j \cdots, v_{l-1}, P+h), \\
E_{l,j}^+(v) &= a_{l,j}^* \oint_{C_{l,j}^{*+}} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_{l-1}(v_{l-1})] g_{l,j}^+(v, v_j \cdots, v_{l-1}, P) \\
&\quad + a_{l,j}^* \oint_{C_{l,j}^{*-}} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [E_{l-1}(v_{l-1}) E_{l-2}(v_{l-2}) \cdots E_j(v_j)] g_{l,j}^-(v, v_j \cdots, v_{l-1}, P), \\
E_{-j,-l}^+(v) &= a_{-j,-l}^* \oint_{C_{-j,-l}^{*+}} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [E_{l-1}(v_{l-1}) E_{l-2}(v_{l-2}) \cdots E_j(v_j)] g_{-j,-l}^+(v, v_j, \cdots, v_{l-1}, P) \\
&\quad + a_{l,j}^* \oint_{C_{l,j}^{*-}} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_{l-1}(v_{l-1})] g_{-j,-l}^-(v, v_j \cdots, v_{l-1}, P),
\end{aligned}$$

$$\begin{aligned}
f_{j,l}^\pm(v, v_j \cdots, v_{l-1}, P+h) &= \frac{[v - v_{l-1} + (P+h)_{j,l} + \frac{l-1}{2} - 1][1]}{[v - v_{l-1} + \frac{l-1}{2}][(P+h)_{j,l} - 1]} \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m + (P+h)_{j,m+1} - \frac{1}{2}][1]}{[v_{m+1} - v_m \pm \frac{1}{2}][(P+h)_{j,m+1}]}, \\
f_{-l,-j}^\pm(v, v_j \cdots, v_{l-1}, P+h) &= \frac{[v - v_j + (P+h)_{-l,-j} - \frac{j}{2} - \eta - 1 + \delta_{l,0}][1]}{[v - v_j - \frac{j}{2} - \eta][(P+h)_{-l,-j} - 1 + \delta_{l,0}]} \prod_{m=j}^{l-2} \frac{[v_m - v_{m+1} + (P+h)_{-l,-(m+1)} - \frac{1}{2} + \delta_{l,0}][1]}{[v_m - v_{m+1} \pm \frac{1}{2}][(P+h)_{-l,-(m+1)} + \delta_{l,0}]}, \\
g_{l,j}^\pm(v, v_j \cdots, v_{l-1}, P) &= \frac{[v - v_{l-1} + \frac{l-1-c}{2} + 1 - P_{j,l}]^*[1]^*}{[v - v_{l-1} + \frac{l-1-c}{2}]^*[P_{j,l} - 1]^*} \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}]^*[1]^*}{[v_{m+1} - v_m \pm \frac{1}{2}]^*[P_{j,m+1} - 1]^*}, \\
g_{-j,-l}^\pm(v, v_j, \cdots, v_{l-1}, P) &= \frac{[v - v_j - \frac{j+c}{2} - \eta + 1 - P_{-l,-j} - \delta_{l,0}]^*[1]^*}{[v - v_j - \frac{j+c}{2} - \eta]^*[P_{-l,-j} - 1 + \delta_{l,0}]^*} \prod_{m=j}^{l-2} \frac{[v_m - v_{m+1} - P_{-l,-(m+1)} + \frac{1}{2} - \delta_{l,0}]^*[1]^*}{[v_m - v_{m+1} \pm \frac{1}{2}]^*[P_{-l,-(m+1)} - 1 + \delta_{l,0}]^*}.
\end{aligned}$$

$$C_{j,l}^\pm : |pq^l w| < |w_{l-1}| < |q^l w|, \quad |pq^{\pm 1} w_{m+1}| < |w_m| < |q^{\pm 1} w_{m+1}|,$$

$$C_{-l,-j}^\pm : |pq^{-j+1} \xi w| < |w_j| < |q^{-j+1} \xi w|, \quad |pq^{\pm 1} w_m| < |w_{m+1}| < |q^{\pm 1} w_m|,$$

$$C_{l,j}^{*\pm} : |q^{l+c} w| < |w_{l-1}| < |p^{*-1} q^{l+c} w|, \quad |q^{\pm 1} w_{m+1}| < |w_m| < |p^{*-1} q^{\pm 1} w_{m+1}|,$$

$$C_{-j,-l}^{*\pm} : |q^{-j+1+c} \xi w| < |w_j| < |p^{*-1} q^{-j+1+c} \xi w|, \quad |q^{\pm 1} w_m| < |w_{m+1}| < |p^{*-1} q^{\pm 1} w_m|$$

for $j \leq m \leq l - 2$. Here $N + 1 \equiv 0$. The case $l \neq 0$ can be proved in the same way as for $U_{q,p}(A_N^{(1)})$ [18].

The following is a conjectural expression for the half currents of the second type, which we obtained by requiring that the integrand should be single-valued and the vector representation of the L -operator should reproduce the R -matrix.

Conjecture B.2. *i) For $j \prec k \preceq N$,*

$$\begin{aligned}
& E_{-k,j}^+(v) \\
&= a_{-k,j}^* \oint_{C_{-k,j}^{*+}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=k}^N \frac{dw'_m}{2\pi i w'_m} [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_k(v_k) \cdots E_N(v_N) E_N(v'_N) \cdots E_k(v'_k)] \\
&\quad \times g_{-k,j}^+(v, v_j, \dots, v'_k, P) \\
&+ a_{-k,j}^* \frac{[1]^*}{[\frac{1}{2}]^*} \oint_{C_{-k,j}^{*-}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=k}^N \frac{dw'_m}{2\pi i w'_m} [E_k(v'_k) E_{k+1}(v'_{k+1}) \cdots E_N(v'_N) E_N(v_N) \cdots E_k(v_k) \cdots E_k(v_j)] \\
&\quad \times g_{-k,j}^-(v, v_j, \dots, v'_k, P),
\end{aligned}$$

$$\begin{aligned}
& g_{-k,j}^+(v, v_j, \dots, v'_k, P) \\
&= \frac{[v - v'_k - \frac{k+c}{2} - \eta - P_{j,-k} + 1]^* [1]^*}{[v - v'_k - \frac{k+c}{2} - \eta]^* [P_{j,-k} - 1]^*} \prod_{m=k+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^* [1]^*}{[v'_{m-1} - v'_m + \frac{1}{2}]^* [P_{j,-m} - 1]^*} \\
&\times \frac{[v'_N - v_N - P_{j,0} + \frac{1}{2}]^* [1]^*}{[v'_N - v_N - \frac{1}{2}]^* [P_{j,0}]^*} \prod_{m=k+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} + \frac{1}{2}]^* [P_{j,m} - 1]^*} \frac{[v_k - v_{k-1} - P_{j,k} - \frac{1}{2}]^* [1]^*}{[v_k - v_{k-1} + \frac{1}{2}]^* [P_{j,k}]^*} \\
&\times \prod_{m=j+1}^{k-1} \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} + \frac{1}{2}]^* [P_{j,m} - 1]^*}, \\
& g_{-k,j}^-(v, v_j, \dots, v'_k, P) \\
&= \frac{[v - v'_k - \frac{k+c}{2} - \eta - P_{j,-k} + 1]^* [1]^*}{[v - v'_k - \frac{k+c}{2} - \eta]^* [P_{j,-k} - 1]^*} \prod_{m=k+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^* [1]^*}{[v'_{m-1} - v'_m - \frac{1}{2}]^* [P_{j,-m} - 1]^*} \\
&\times \frac{[v'_N - v_N]^*}{[v'_N - v_N + \frac{1}{2}]^*} \frac{[v'_N - v_N - P_{j,0}]^* [1]^*}{[v'_N - v_N - \frac{3}{2}]^* [P_{j,0} - \frac{3}{2}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} - \frac{1}{2}]^* [P_{j,m} - 1]^*}.
\end{aligned}$$

$$\begin{aligned}
C_{-k,j}^{*\pm} &: |q^{-k+1+c} \xi w| < |w'_k| < |p^{*-1} q^{-k+1+c} \xi w|, \\
&|q^{\pm 1} w'_{m-1}| < |w'_m| < |p^{*-1} q^{\pm 1} w'_{m-1}| \quad (k+1 \leq m \leq N), \\
&|q^{\mp 1} w'_N| < |w_N| < |p^{*-1} q^{\mp 1} w'_N|, \quad |q^{\pm 1} w_n| < |w_{n-1}| < |p^{*-1} q^{\pm 1} w_n| \quad (j+1 \leq n \leq N)
\end{aligned}$$

ii) For $j = k \prec N$, we obtain from (B.3) and (B.11),

$$\begin{aligned}
& E_{-j,j}^+(v) \\
&= a_{-j,j}^* \oint_{C_{-j,j}^{*+}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_N(v_N) E_N(v'_N) \cdots E_k(v'_j)] \\
&\quad \times g_{-j,j}^+(v, v_j, \dots, v'_j, P) \\
&+ a_{-j,j}^* \frac{[1]^*}{[\frac{1}{2}]^*} \oint_{C_{-j,j}^{*-}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} [E_j(v'_j) E_{j+1}(v'_{j+1}) \cdots E_N(v'_N) E_N(v_N) \cdots E_k(v_j)] \\
&\quad \times g_{-j,j}^-(v, v_j, \dots, v'_j, P), \\
&g_{-j,j}^+(v, v_j, \dots, v'_j, P) \\
&= \frac{[v - v'_j - \frac{j+c}{2} - \eta - 2P_j + 1]^* [1]^*}{[v - v'_j - \frac{j+c}{2} - \eta]^* [2P_j - 1]^*} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^* [1]^*}{[v'_{m-1} - v'_m + \frac{1}{2}]^* [P_{j,-m} - 1]^*} \\
&\quad \times \frac{[v'_N - v_N - P_{j,0} + \frac{1}{2}]^* [1]^*}{[v'_N - v_N - \frac{1}{2}]^* [P_{j,0}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} + \frac{1}{2}]^* [P_{j,m} - 1]^*} \frac{[v - v_j + \frac{-j+1+c}{2} - \eta + 1]^*}{[v - v_j + \frac{-j+1+c}{2} - \eta]^*}, \\
&g_{-j,j}^-(v, v_j, \dots, v'_j, P) \\
&= \frac{[v - v'_j - \frac{j+c}{2} - \eta - 2P_j + 2]^* [1]^*}{[v - v'_j - \frac{j+c}{2} - \eta]^* [2P_j - 1]^*} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^* [1]^*}{[v'_{m-1} - v'_m - \frac{1}{2}]^* [P_{j,-m} - 1]^*} \\
&\quad \times \frac{[v'_N - v_N]^*}{[v'_N - v_N + \frac{1}{2}]^*} \frac{[v'_N - v_N - P_{j,0}]^* [1]^*}{[v'_N - v_N - \frac{3}{2}]^* [P_{j,0} - \frac{3}{2}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} - \frac{1}{2}]^* [P_{j,m} - 1]^*}.
\end{aligned}$$

$$\begin{aligned}
C_{-j,j}^{\pm} : & \quad |q^{-j+1+c} \xi w| < |w'_j| < |p^{*-1} q^{-j+1+c} \xi w|, \quad |q^{\pm 1} w'_{m-1}| < |w'_m| < |p^{*-1} q^{\pm 1} w'_{m-1}|, \\
& \quad |q^{\mp 1} w'_N| < |w_N| < |p^{*-1} q^{\mp 1} w'_N|, \quad |q^{\pm 1} w_m| < |w_{m-1}| < |p^{*-1} q^{\pm 1} w_m|
\end{aligned}$$

for $j+1 \leq m \leq N$ with

$$\begin{aligned}
& |q^{-j+1+c} \xi w| < |w_j| < |p^{*-1} q^{-j+1+c} \xi w| \quad \text{for } C_{-j,j}^{*+}, \\
& |q^{-3} w'_N| < |w_N| < |p^{*-1} q^{-3} w'_N| \quad \text{for } C_{-j,j}^{*-}.
\end{aligned}$$

iii) For $k \prec j \preceq N$, we obtain from (B.5)

$$\begin{aligned}
& E_{-k,j}^+(v) \\
&= a_{-k,j}^* \oint_{C_{-k,j}^{*+}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=k}^N \frac{dw'_m}{2\pi i w'_m} [E_j(v_j) E_{j+1}(v_{j+1}) \cdots E_N(v_N) E_N(v'_N) \cdots E_j(v'_j) \cdots E_k(v'_k)] \\
&\quad \times g_{-k,j}^+(v, v_j, \dots, v'_k, P) \\
&+ a_{-k,j}^* \frac{[1]^*}{[\frac{1}{2}]^*} \oint_{C_{-k,j}^{*-}} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=k}^N \frac{dw'_m}{2\pi i w'_m} [E_k(v'_k) E_{k+1}(v'_{k+1}) \cdots E_j(v'_j) \cdots E_N(v'_N) E_N(v_N) \cdots E_k(v_j)] \\
&\quad \times g_{-k,j}^-(v, v_j, \dots, v'_k, P), \\
&g_{-k,j}^+(v, v_j, \dots, v'_k, P), \\
&= \frac{[v - v'_k - \frac{k+c}{2} - \eta - P_{j,-k} + 1]^* [1]^*}{[v - v'_k - \frac{k+c}{2} - \eta]^* [P_{j,-k} - 1]^*} \prod_{m=k+1}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^* [1]^*}{[v'_{m-1} - v'_m + \frac{1}{2}]^* [P_{j,-m} - 1]^*} \\
&\quad \times \frac{[v'_N - v_N - P_{j,0} + \frac{1}{2}]^* [1]^*}{[v'_N - v_N - \frac{1}{2}]^* [P_{j,0}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} + \frac{1}{2}]^* [P_{j,m} - 1]^*}, \\
&g_{-k,j}^-(v, v_j, \dots, v'_k, P) \\
&= \frac{[v - v'_k - \frac{k+c}{2} - \eta - P_{j,-k} + 1]^* [1]^*}{[v - v'_k - \frac{k+c}{2} - \eta]^* [P_{j,-k} - 1]^*} \prod_{\substack{m=k+1 \\ m \neq j}}^N \frac{[v'_{m-1} - v'_m - P_{j,-m} + \frac{1}{2}]^* [1]^*}{[v'_{m-1} - v'_m - \frac{1}{2}]^* [P_{j,-m} - 1]^*} \frac{[v'_{j-1} - v'_j - 2P_j + \frac{3}{2}]^* [1]^*}{[v'_{j-1} - v'_j - \frac{1}{2}]^* [2P_j - 1]^*} \\
&\quad \times \frac{[v'_N - v_N]^*}{[v'_N - v_N + \frac{1}{2}]^*} \frac{[v'_N - v_N - P_{j,0}]^* [1]^*}{[v'_N - v_N - \frac{3}{2}]^* [P_{j,0} - \frac{3}{2}]^*} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} - P_{j,m} + \frac{1}{2}]^* [1]^*}{[v_m - v_{m-1} - \frac{1}{2}]^* [P_{j,m} - 1]^*}. \\
&C_{-k,j}^{*\pm} : |q^{-k+1+c} \xi w| < |w'_k| < |p^{*-1} q^{-k+1+c} \xi w|, \\
&\quad |q^{\pm 1} w'_{m-1}| < |w'_m| < |p^{*-1} q^{\pm 1} w'_{m-1}| \quad (k+1 \leq m \leq N), \\
&\quad |q^{\mp 1} w'_N| < |w_N| < |p^{*-1} q^{\mp 1} w'_N|, \quad |q^{\pm 1} w_n| < |w_{n-1}| < |p^{*-1} q^{\pm 1} w_n| \quad (j+1 \leq n \leq N), \\
&\text{in addition, for } C_{-k,j}^{*-}
\end{aligned}$$

$$|q^{-3} w'_N| < |w_N| < |p^{*-1} q^{-3} w'_N|.$$

vi) For $j = k = N$, we obtain from (B.11) and (B.12),

$$\begin{aligned}
E_{-N,N}^+(v) &= a_{-N,N}^* \oint_{C_{-NN}^{*+}} \frac{dw_N}{2\pi i w_N} \frac{dw'_N}{2\pi i w'_N} [E_N(v_N) E_N(v'_N)] g_{-N,N}^+(v, v_N, v'_N, P) \\
&\quad + a_{-N,N}^* \oint_{C_{-NN}^{*-}} \frac{dw_N}{2\pi i w_N} \frac{dw'_N}{2\pi i w'_N} [E_N(v'_N) E_N(v_N)] g_{-N,N}^-(v, v_N, v'_N, P), \\
g_{-N,N}^+(v, v_N, v'_N, P) &= \frac{[v - v'_N - \frac{N+c}{2} - \eta - 2P_N + \frac{3}{2}]^* [1]^*}{[v - v'_N - \frac{N+c}{2} - \eta]^* [2P_N - 1]^*} \\
&\quad \times \frac{[v'_N - v_N - P_{N,0} + 1]^* [1]^*}{[v'_N - v_N - \frac{1}{2}]^* [P_{N,0} - \frac{3}{2}]^*} \frac{[v - v_N - \frac{N+c}{2} - \eta + \frac{1}{2}]^*}{[v - v_N - \frac{N+c}{2} - \eta]^*}, \\
g_{-N,N}^-(v, v_N, v'_N, P) &= \frac{[v - v'_N - \frac{N+c}{2} - \eta - 2P_N + \frac{3}{2}]^* [1]^*}{[v - v'_N - \frac{N+c}{2} - \eta]^* [2P_N - 1]^*} \\
&\quad \times \frac{[v'_N - v_N - P_{N,0} + 1]^* [1]^*}{[v'_N - v_N + \frac{1}{2}]^* [P_{N,0} - \frac{3}{2}]^*} \frac{[v - v_N - \frac{N+c}{2} - \eta + \frac{1}{2}]^*}{[v - v_N - \frac{N+c}{2} - \eta]^*}.
\end{aligned}$$

$$C_{-N,N}^{*\pm} : |q^{-N+1+c}\xi w| < |w_N|, |w'_N| < |p^{*-1}q^{-N+1+c}\xi w|, \quad |q^{\mp 1}w'_N| < |w_N| < |p^{*-1}q^{\mp 1}w'_N|.$$

Conjecture B.3. i) For $j \prec k \preceq N$,

$$\begin{aligned}
&F_{k,-j}^+(v) \\
&= a_{k,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{k-j}^+} \prod_{m=k}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} [F_k(v_k) F_{k+1}(v_{k+1}) \cdots F_N(v_N) F_N(v'_N) \cdots F_k(v'_k) \cdots F_j(v'_j)] \\
&\quad \times f_{k,-j}^+(v, v_k, \dots, v'_j, P+h) \\
&\quad + a_{k,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{k-j}^-} \prod_{m=k}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} [F_j(v'_j) F_{j+1}(v'_{j+1}) \cdots F_k(v'_k) \cdots F_N(v'_N) F_N(v_N) \cdots F_k(v_k)] \\
&\quad \times f_{k,-j}^-(v, v_k, \dots, v'_j, P+h), \\
&f_{k,-j}^+(v, v_k, \dots, v'_j, P+h) \\
&= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + (P+h)_{k,-j}][1]}{[v - v'_j - \frac{j}{2} - \eta][P_{k,-j} - 1]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{k,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m - \frac{1}{2}][(P+h)_{k,-m}]} \\
&\quad \times \frac{[v'_N - v_N]}{[v'_N - v_N + \frac{1}{2}]} \frac{[v'_N - v_N + (P+h)_{k,0} - 2][1]}{[v'_N - v_N - \frac{3}{2}][(P+h)_{k,0} - \frac{1}{2}]} \prod_{m=k+1}^N \frac{[v_m - v_{m-1} + (P+h)_{k,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} - \frac{1}{2}][(P+h)_{k,m}]}, \\
&f_{k,-j}^-(v, v_k, \dots, v'_j, P+h) \\
&\quad \times \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + (P+h)_{k,-j}][1]}{[v - v'_j - \frac{j}{2} - \eta][P_{k,-j} - 1]} \prod_{\substack{m=j+1 \\ \neq k}}^N \frac{[v'_{m-1} - v'_m + (P+h)_{k,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m + \frac{1}{2}][(P+h)_{k,-m}]} \\
&\quad \times \frac{[v'_{k-1} - v'_k + 2(P+h)_k - \frac{3}{2}][1]}{[v'_{k-1} - v'_k + \frac{1}{2}][2(P+h)_k - 1]} \frac{[v'_N - v_N + (P+h)_{k,0} - \frac{3}{2}][1]}{[v'_N - v_N - \frac{1}{2}][(P+h)_{k,0} - 2]} \\
&\quad \times \prod_{m=k+1}^N \frac{[v_m - v_{m-1} + (P+h)_{k,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} + \frac{1}{2}][(P+h)_{k,m}]}.
\end{aligned}$$

$$C_{k,-j}^{\pm} : |pq^{-j+1}\xi w| < |w'_j| < |q^{-j+1}\xi w|, \quad |pq^{\mp 1}w'_{m-1}| < |w'_m| < |q^{\mp 1}w'_{m-1}| \quad (j+1 \leq m \leq N),$$

$$|pq^{\pm 1}w'_N| < |w_N| < |q^{\pm 1}w'_N|, \quad |pq^{\mp 1}w_n| < |w_{n-1}| < |q^{\mp 1}w_n| \quad (k+1 \leq n \leq N),$$

in addition, for $C_{k,-j}^+$

$$|pq^{-3}w'_N| < |w_N| < |q^{-3}w'_N|.$$

ii) For $j = k \prec N$, we obtain from (B.3) and (B.11),

$$\begin{aligned} & F_{j,-j}^+(v) \\ &= a_{j,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{j,-j}^+} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} [F_j(v_j) F_{j+1}(v_{j+1}) \cdots F_N(v_N) F_N(v'_N) \cdots F_j(v'_j)] \\ & \quad \times f_{j,-j}^+(v, v_j, \dots, v'_j, P+h) \\ &+ a_{j,-j} \oint_{C_{j,-j}^-} \prod_{m=j}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} [F_j(v'_j) F_{j+1}(v'_{j+1}) \cdots F_N(v'_N) F_N(v_N) \cdots F_j(v_j)] \\ & \quad \times f_{j,-j}^-(v, v_j, \dots, v'_j, P+h), \\ & f_{j,-j}^+(v, v_j, \dots, v'_j, P+h) \\ &= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + 2(P+h)_j - 1][1]}{[v - v'_j - \frac{j}{2} - \eta][2(P+h)_j - 3]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{j,-m} - \frac{3}{2}][1]}{[v'_{m-1} - v'_m - \frac{1}{2}][(P+h)_{j,-m} - 1]} \\ & \quad \times \frac{[v'_N - v_N]}{[v'_N - v_N + \frac{1}{2}]} \frac{[v'_N - v_N + (P+h)_{j,0} - 3][1]}{[v'_N - v_N - \frac{3}{2}][(P+h)_{j,0} - \frac{3}{2}]} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} + (P+h)_{j,m} - \frac{3}{2}][1]}{[v_m - v_{m-1} - \frac{1}{2}][(P+h)_{j,m} - 1]}, \\ & f_{j,-j}^-(v, v_j, \dots, v'_j, P+h) \\ &= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + 2(P+h)_j - 2][1]}{[v - v'_j - \frac{j}{2} - \eta][2(P+h)_j - 3]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{j,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m + \frac{1}{2}][(P+h)_{j,-m}]} \\ & \quad \times \frac{[v'_N - v_N + (P+h)_{j,0} - \frac{3}{2}][1]}{[v'_N - v_N - \frac{1}{2}][(P+h)_{j,0} - 2]} \prod_{m=j+1}^N \frac{[v_m - v_{m-1} + (P+h)_{j,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} + \frac{1}{2}][(P+h)_{j,m}]} \\ & \quad \times \frac{[v - v'_j - \frac{3j-2}{2} - 2\eta]}{[v - v'_j - \frac{3j-2}{2} - 2\eta - 1]}. \end{aligned}$$

$$C_{j,-j}^{\pm} : |pq^{-j+1}\xi w| < |w'_j| < |q^{-j+1}\xi w|, \quad |pq^{\mp 1}w'_{m-1}| < |w'_m| < |q^{\mp 1}w'_{m-1}|,$$

$$|pq^{\pm 1}w'_N| < |w_N| < |q^{\pm 1}w'_N|, \quad |q^{\mp 1}w_m| < |w_{m-1}| < |q^{\mp 1}w_m|$$

for $j+1 \leq m \leq N$ with

$$|pq^{-3}w'_N| < |w_N| < |q^{-3}w'_N| \quad \text{for } C_{j,-j}^+,$$

$$|pq^{-3j-5}\xi^2 w| < |w'_j| < |q^{-3j-5}\xi^2 w| \quad \text{for } C_{j,-j}^-.$$

iii) $k \prec j \preceq N$

$$\begin{aligned}
& F_{k,-j}^+(v) \\
&= a_{k,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{k,-j}^+} \prod_{m=k}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} [F_k(v_k) F_{k+1}(v_{k+1}) \cdots F_j(v_j) \cdots F_N(v_N) F_N(v'_N) \cdots F_j(v'_j)] \\
&\quad \times f_{k,-j}^+(v, v_k, \dots, v'_j, P+h) \\
&+ a_{k,-j} \frac{[1]}{[\frac{1}{2}]} \oint_{C_{k,-j}^-} \prod_{m=k}^N \frac{dw_m}{2\pi i w_m} \prod_{m=j}^N \frac{dw'_m}{2\pi i w'_m} [F_j(v'_j) F_{j+1}(v'_{j+1}) \cdots F_N(v'_N) F_N(v_N) \cdots F_j(v_j) \cdots F_k(v_k)] \\
&\quad \times f_{k,-j}^-(v, v_k, \dots, v'_j, P+h), \\
& f_{k,-j}^+(v, v_k, \dots, v'_j, P+h) \\
&= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + (P+h)_{k,-j}][1]}{[v - v'_j - \frac{j}{2} - \eta][P_{k,-j} - 1]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{k,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m - \frac{1}{2}][(P+h)_{k,-m}]} \\
&\quad \times \frac{[v'_N - v_N]}{[v'_N - v_N + \frac{1}{2}]} \frac{[v'_N - v_N + (P+h)_{k,0} - 2][1]}{[v'_N - v_N - \frac{3}{2}][(P+h)_{k,0} - \frac{1}{2}]} \prod_{\substack{m=k+1 \\ \neq j}}^N \frac{[v_m - v_{m-1} + (P+h)_{k,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} - \frac{1}{2}][(P+h)_{k,m}]} \\
&\quad \times \frac{[v_j - v_{j-1} + (P+h)_{k,j} + \frac{1}{2}][1]}{[v_j - v_{j-1} - \frac{1}{2}][(P+h)_{k,j}]}, \\
& f_{k,-j}^-(v, v_k, \dots, v'_j, P+h) \\
&= \frac{[v - v'_j - \frac{j}{2} - \eta - 1 + (P+h)_{k,-j}][1]}{[v - v'_j - \frac{j}{2} - \eta][P_{k,-j} - 1]} \prod_{m=j+1}^N \frac{[v'_{m-1} - v'_m + (P+h)_{k,-m} - \frac{1}{2}][1]}{[v'_{m-1} - v'_m + \frac{1}{2}][(P+h)_{k,-m}]} \\
&\quad \times \frac{[v'_N - v_N + (P+h)_{k,0} - \frac{3}{2}][1]}{[v'_N - v_N - \frac{1}{2}][(P+h)_{k,0} - 2]} \prod_{m=k+1}^N \frac{[v_m - v_{m-1} + (P+h)_{k,m} - \frac{1}{2}][1]}{[v_m - v_{m-1} + \frac{1}{2}][(P+h)_{k,m}]}.
\end{aligned}$$

$$\begin{aligned}
C_{k,-j}^\pm &: |pq^{-j+1}\xi w| < |w'_j| < |q^{-j+1}\xi w|, \quad |pq^{\mp 1}w'_{m-1}| < |w'_m| < |q^{\mp 1}w'_{m-1}| \quad (j+1 \leq m \leq N), \\
&|pq^{\pm 1}w'_N| < |w_N| < |q^{\pm 1}w'_N|, \quad |pq^{\mp 1}w_n| < |w_{n-1}| < |q^{\mp 1}w_n| \quad (k+1 \leq n \leq N),
\end{aligned}$$

in addition, for $C_{k,-j}^+$

$$|pq^{-3}w'_N| < |w_N| < |q^{-3}w'_N|.$$

vi) $j = k = N$

$$\begin{aligned}
F_{N,-N}^+(v) &= a_{N,-N} \oint_{C_{N,-N}^+} \frac{dw_N}{2\pi i w_N} \frac{dw'_N}{2\pi i w'_N} [F_N(v'_N) F_N(v_N)] f_{N,-N}^+(v, v_N, v'_N, P+h) \\
&\quad + a_{N,-N} \oint_{C_{N,-N}^-} \frac{dw_N}{2\pi i w_N} \frac{dw'_N}{2\pi i w'_N} [F_N(v_N) F_N(v'_N)] f_{N,-N}^-(v, v_N, v'_N, P+h), \\
f_{N,-N}^+(v, v_N, v'_N, P+h) &= \frac{[v - v'_N - \frac{N}{2} - \eta + 2(P+h)_N - \frac{5}{2}][1]}{[v - v'_N - \frac{N}{2} - \eta][2(P+h)_N - 3]} \\
&\quad \times \frac{[v'_N - v_N + (P+h)_{N,0} - 2][1]}{[v'_N - v_N - \frac{1}{2}][(P+h)_{N,0} - \frac{3}{2}]} \frac{[v - v_N - \frac{N-1}{2} - \eta + \frac{1}{2}]}{[v - v_N - \frac{N-1}{2} - \eta]}, \\
f_{N,-N}^-(v, v_N, v'_N, P+h) &= \frac{[v - v'_N - \frac{N}{2} - \eta + 2(P+h)_N - \frac{5}{2}][1]}{[v - v'_N - \frac{N}{2} - \eta][2(P+h)_N - 3]} \\
&\quad \times \frac{[v'_N - v_N + (P+h)_{N,0} - 2][1]}{[v'_N - v_N + \frac{1}{2}][(P+h)_{N,0} - \frac{3}{2}]} \frac{[v - v_N - \frac{N-1}{2} - \eta + \frac{1}{2}]}{[v - v_N - \frac{N-1}{2} - \eta]}.
\end{aligned}$$

$$C_{N,-N}^\pm : |pq^{-N+1}\xi w| < |w_N|, |w'_N| < |q^{-N+1}\xi w|, \quad |pq^{\mp 1}w'_N| < |w_N| < |q^{\mp 1}w'_N|.$$

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